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ON THE FOUNDATIONS OF RELATIVITY THEORY

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The system of axioms which we propose is based on the definition of space-time given by A. D. Aleksandrov [1], [2]. According to this definition, space-time is the set of all events in the universe, abstracted from all their characteristics except those determined by relations of action of some events on others. In other words, the space-time structure of the universe is nothing else but its cause and effect structure taken in the corresponding abstraction.

1. Let V be some nonempty set of points—a universe of events—on which a partial order $\leqslant$ is given—the relation of precedence or action between events. Let

$$P_x = \{ y \in V : x \leqslant y \}, \qquad P_x^- = \{ y \in V : y \leqslant x \}.$$

The set P_x (P_x^-) is interpreted as the set of events which can be influenced by (respectively, can influence) the event x.

1.1. *Axioms of order.*

A_1. The system of sets $\{ P_x \}$ gives a nontrivial partial order on V, i.e. 1) any point x from V belongs to P_x, 2) $P_x \neq \{ x \}$, and 3) if $y \in P_x$, then $P_y \subset P_x$.

A_2. No interval $P_x \cap P_y^-$ reduces to a pair of points x, y, i.e. $P_x \cap P_y^- \neq \{ x, y \}$.

Axiom A_2 postulates that the action from x to y is transmitted through intermediate events and not stepwise.

1.2. *Axiom of topology.* Let $\mathscr{B}$ be the family of all sets of the form $(P_x \setminus P_y^-) \setminus A_{xy}$, where A_{xy} is the union of all linearly ordered intervals $P_a \cap P_b^- \subset P_x \cap P_y^-$, $b \in P_a$, such that either $a = x$ or $b = y$. Consider on V the topology $\mathscr{T}_\leqslant$ with prebasis $\mathscr{B}$.

A_3. $\langle V, \mathscr{T}_\leqslant \rangle$ is a simply connected four-dimensional locally compact Hausdorff space.

1.3. *Axiom of space-time homogeneity.*

A_4. The commutative group T of homeomorphisms of V onto itself acts transitively on the space $\langle V, \mathscr{T}_\leqslant \rangle$, so that for any $x \in V$ and $t \in T$ we have $t(P_x) = P_{t(x)}$.

THEOREM 1 [3]. *If axioms* A_3 *and* A_4 *hold, then* $\langle V, \mathscr{T}_\leqslant \rangle$ *can be equipped with an affine structure in such away that* $\mathscr{T}_\leqslant$ *will be equivalent to the natural topology of the four-dimensional affine space, and* T *will be isomorphic to a group of parallel translations.*

1.4. *Axioms of connection between order and topology.*

A_5. For any $x, y \in V$ the intervals $P_x \cap P_y^-$ are compact.

A_6. All sets P_x are closed: $P_x = \overline{P}_x$.

A_7. Every set P_x has interior points, int $P_x \neq \varnothing$.

Axiom A_5 means that the velocity of transmission of any action is limited; A_6 considers the existence of an action with a fundamental velocity which is a limit for other actions

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and a bound for their velocity of propagation. We can give the following interpretation of A_7. From a certain instant of time, the action propagates continuously in all directions at least in one point of 3-space. Besides, it follows from this axiom that the prebasis $\mathscr{B}$ is not empty.

1.5. *Axiom of connection between order and movement.* Let G be the group of all order-preserving bijections of V onto itself, i.e. for any $g \in G$ and $x \in V$ we have $g(P_x) = P_{g(x)}$. We denote by G_x the subgroup of G consisting of bijections $g \in G$ such that $g(x) = x$. Since the topology $\mathscr{T}_\leqslant$ is an order-determined one, the group G consists of continuous mappings and, since they are bijective, of homeomorphisms.

A_8. The group G_x acts transitively on $\partial P_x \setminus \{x\}$, where ∂P_x is the boundary of the set P_x.

Axiom A_8 can be called the axiom of isotropy of the 3-space, as it speaks about the equivalence between all directions of the physical space.

1.6. THEOREM 2. *Assume that the system of axioms $\langle A_1 - A_8 \rangle$ holds. Then $\langle V, \mathscr{T}_\leqslant \rangle$ can be identified with a four-dimensional affine space, and the group T can be identified with its group of parallel translations. In V one can introduce Cartesian coordinates $x = (x_0, x_1, x_2, x_3)$ such that*

$$P_x = \left\{ y \in V \colon (x_0 - y_0)^2 - \sum_{i=1}^{3} (x_i - y_i)^2 \geqslant \text{ and } x_0 \leqslant y_0 \right\},$$

i.e. $\{P_x\}$ is a system of equal and parallel closed solid elliptic cones, and the group G is isomorphic to the Poincaré group including the dilations $x \to \lambda x$, $\lambda > 0$, and excluding the reflections $x \to (-x_0, x_1, x_2, x_3)$.

PROOF. The affineness of the structure of the space V is proved in Theorem 1. Then, we use the following theorem formulated by A. D. Aleksandrov and proved in [4].

THEOREM. *If in a finite-dimensional affine space V an order invariant relative to parallel translation is given, such that all P_x are closed, and all intervals $P_x \cap P_y^-$ are bounded and do not reduce to $\{x, y\}$, then $\{P_x\}$ is a system of equal and parallel convex cones.*

It follows from A_5 that the cones P_x have an acute vertex, i.e. they do not contain any straight line. Thus, the group G preserves the system of cones $\{P_x\}$. If P_x is not a quasi-cylinder, then all transformations from G are affine ([5], Theorem 3). If P_x is a quasi-cylinder, i.e. $P_x = L_1 \times \cdots \times L_k \times K_x$, where the L_i are rays issued from x and K_x is a closed convex cone which cannot be represented as a direct product of a ray by a cone with a smaller dimension, then P_x has distinguished extreme rays L_i which cannot be translated with the help of the group G_x into any given generator of the cone ∂P_x. The latter contradicts axiom A_8. So, all transformations of G are affine and the subgroup G_x acts transitively on the generators of the cone ∂P_x, as follows from A_8. According to a theorem of Busemann ([6], p. 34), the cone ∂P_x is elliptic. Hence, G is the Poincaré group plus the dilations $x \to \lambda x$, $\lambda > 0$, and without the reflections $x \to (-x_0, x_1, x_2, x_3)$. The theorem is proved.

2. Let us drop the requirement that the topology be determined by an order, and let us introduce the axiom

A_3^*. V is a simply connected four-dimensional locally compact Hausdorff space.

THEOREM 3. *If axioms $\langle A_1, A_2, A_3^*, A_4 - A_8 \rangle$ hold, then the statement of Theorem 2 is correct.*

The proof remains identical if we note that, on the basis of the continuity theorem from [2], §3.1, Theorem 1, all $g \in G$ turns out to be isomorphisms.

3. In the system of axioms stated above the most unsatisfactory is the requirement of the commutativity of the group T. We shall show ho to avoid it. For the sake of brevity, we shall consider the space-time as an ordered group $V = T$. Rejection of commutativity makes it necessary to distinguish the right and the left product of elements of the group V. For instance, in [7] de Sitter's space-time axiomatics is described in terms of a left-invariant order which is not a bi-invariant one. Thus, we come necessarily of the study of bi-invariant orders on groups.

Let $\mathscr{T}_{\leqslant}$ be the topology on V described at the beginning of the article. Let us introduce the following axioms.

B_1. $\langle V, \mathscr{T}_{\leqslant} \rangle$ is a connected locally-compact four-dimensional topological Hausdorff group.

B_2. The order $\{P_x\}$ on V is bi-invariant, i.e. $P \cdot P \subset P$, where $P = P_e$, e being the unit of the group V, $P_x = x \cdot P = P \cdot x$ for all $x \in V$.

B_3. There exists a point $x \in \partial P \setminus \{e\}$ such that the set

$$M_x = \overline{\bigcup_{e, x \in \partial P_y} P_y}$$

is a maximal subsemigroup, i.e. if $M_x \subset H \subset V$, where H is a subsemigroup, then $H = M_x$ or $H = V$.

It follows from B_2 that M_x is a normal subsemigroup, i.e. for any $a \in V$ we have $a \cdot M_x = M_x \cdot a$. Let $\mathscr{K}$ be the family of all sets of the type $\bigcap_{\alpha \in A} x_\alpha \cdot M_\alpha$, where $x_\alpha \in V$ and $M_\alpha \in M$, M being the family of all maximal subsemigroups of V, and A is a set of indices (arbitrary).

We shall call the sets of the family $\mathscr{K}$ convex sets.

B_4. The topology $\mathscr{T}_{\leqslant}$ is convex, i.e. there exists a basis of neighborhoods of the unit of the group, consisting of neighborhoods whos closures are convex sets.

THEOREM 4. *Let the system of axioms* $\langle \mathrm{A}_1, \mathrm{A}_2, \mathrm{B}_1\text{–}\mathrm{B}_4, \mathrm{A}_5\text{–}\mathrm{A}_8 \rangle$ *hold. Then the group* $\langle V, \mathscr{T}_{\leqslant} \rangle$ *is isomorphic to the* 4-*dimensional* (*vector*) *space* $\mathbf{R}^4$ *with the natural topology,* $\{P_x\}$ *is a system of equal and parallel closed solid elliptic cones in* $\mathbf{R}^4$ (*if* V *is identified with* $\mathbf{R}^4$), *and the group* G *is isomorphic to the Poincaré group including the dilations* $x \to \lambda x$, $\lambda > 0$, *and excluding the reflections* $x \to (-x_0, x_1, x_2, x_3)$.

PROOF. It follows from B_1–B_4 and [8], Theorem 2, that $\langle V, \mathscr{T}_{\leqslant} \rangle \cong \mathbf{R}^4$. Identifying V with $\mathbf{R}^4$, we get an invariant order P in $\mathbf{R}^4$ which satisfies A_1, A_2 and A_5–A_8. The rest is analogous to the proof of Theorem 2.

4. It seems probable that for connected simply connected manifolds of general relativity theory which admit a simple-transitive group of motions (they are classified in [9]) a unique axiomatization is possible, similar to the one considered in this article (see also in [7] the space-time axiomatics of de Sitter). From the point of view of the fundamental role played by causality in the material world which surrounds us, this approach seems more natural than Einstein's equations used as a starting point. Obviously, in order to construct this axiomatics, we need the theory of ordered Lie groups, many questions of which were worked out these last years.

4.1. Let V be a simply connected real Lie group on which a bi-invariant order P is given, such that $P \cap P^{-1} = \{e\}$ and $\overline{P} = P$. As was shown by Vinberg [10], the contingency K of the set P at the unit of the group is a closed convex cone in the Lie algebra v of the group V. Its image is contained in P under the exponential mapping. Clearly, K does not contain any straight line. In this connection, K must be invariant with respect to the action of all operators of the adjoint reresentation of V, in particular, with respect to the action of $e^{\operatorname{ad} x}$ type operators.

THEOREM 5. *If a bi-invariant order on a four-dimensional noncommutative Lie group V is such that the contingency K contains interior points, then either $V = G_4$III with $q = 0$, or G_4VII or G_4VIII.*

Here we used the notation of groups from [9].

Thus, the condition of commutativity of the group in the axiomatics of special relativity theory is weakened until exponentiality (a group is called exponential if the exponential mapping is a diffeomorphism of the algebra on the group). Indeed, the groups G_4VII and G_4VIII are unsolvable and, hence, nonexponential, and the group G_4III with $q = 0$ (hereafter we shall denote it simply $G_4\mathrm{III}_0$) is solvable but nonexponential.

Generally speaking, among all 4-dimensional Lie groups, four groups are nonexponential, and so are all groups of the one-parameter family $G_4\mathrm{VI}_2(k, l)$ with $k = 0$.

4.2. Let us call an invariant convex cone with interior points and acute vertex a *causal cone*. Theorem 5 follows in an obvious way from the next statement.

THEOREM 6. *Let the noncommutative Lie algebra v have a causal cone. Then v is $g_4\mathrm{III}_0$ or g_4VII or g_4VIII.*

To prove Theorem 6 we shall use some statements which have an independent interest.

ASSERTION 1. *If the noncommutative Lie algebra v is a semidirect sum of a nilpotent ideal n and an abelian subalgebra such that the center $c(n)$ is not contained in $c(v)$, then v does not contain a causal cone.*

PROOF. Assume that such a cone K exists. Let us take $x \in c(n) \setminus c(v)$. Since $\operatorname{int} K \neq \varnothing$, there exists $y \in K$ such that $\operatorname{ad}_x(y) \neq 0$. Then $e^{\operatorname{ad}_{\lambda x}}(y) = y + \lambda \cdot \operatorname{ad}_x(y)$, $\lambda \in \mathbf{R}$; hence K contains the straight line with the directing vector $\operatorname{ad}_x(y)$.

Similarly we prove

ASSERTION 2. *If a nilpotent Lie algebra has a causal cone, then it is commutative.*

It turns out that causal cones in the algebras $g_4\mathrm{III}_0$, g_4VII and g_4VIII are determined by a bi-invariant metric on the corresponding groups. Hence, from the theorem follows the

COROLLARY. *A bi-invariant metric of signature $+2$ (or -2) exists on a noncommutative Lie group V if V is $G_4\mathrm{III}_0$ or G_4VII or G_4VIII.*

5. Thus, Theorem 5 states that the Minkowski space and the spaces $G_4\mathrm{III}_0$, G_4VII and G_4VIII possess a higher consistency between causal and group properties than other ordered 4-dimensional Lie groups. From this point of view, the choice of the Minkowski space does not require further comments.

The group G_4VIII is the universal covering of a conformal space. A. D. Aleksandrov has stated in Theorem 4 of [11] that the geometry of this space is determined both by its local and global causal structures. By introducing a corresponding bi-invariant metric on G_4VIII we get an important cosmological model (see [12]).

Introducing a bi-invariant metric given in a basis of left-invariant vector fields X_i, $i = 1, 2, 3, 4$ (the basis is taken from [9], Russian p. 73), by the coefficients $g_{14} = g_{41} = 1$, $g_{22} = g_{33} = -1$, other $g_{ij} = 0$, we get a space-time with a subgroup $G_3\mathrm{II}$ of Bianchi type acting on an isotropic hypersurface. Since the isotropic vector field X_1 is covariantly constant, this space-time belongs to the class of *pp*-waves (see [13], §21.5).

We can introduce a bi-invariant metric on $G_4\mathrm{VII} = SL(2, \mathbf{R}) \times \mathbf{R}$ as a direct sum of the Killing metric on $SL(2, \mathbf{R})$ and the Euclidean metric on $\mathbf{R}$; $SL(2, \mathbf{R})$ acts on a time-like hypersurface.

We do not know of any work where the choice of the last two spaces is mentioned.

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