SEMIGROUPS OF THE BASIC AFFINE LIE GROUP AND THEIR AUTOMORPHISMS

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In recent years the theory of semigroups of Lie groups has been intensively studied. The problem of calculating the automorphisms of connected subsemigroups generates particular interest. For Abelian Lie groups this problem has been completely solved by A.D. Aleksandrov [1]. However, for noncommutative Lie groups, no notable results have been obtained yet (the basic affine Lie group [2, 3] and 3-dimensional Lie groups [4] being exceptions).

In this article we find the automorphisms of connected subsemigroups with quasicontingency of the form $L \times K$ (see Definition 1 below) for the case of the basic affine Lie group. This completely solves the problem of computing the automorphisms of subsemigroups of the basic affine Lie group.

The basic affine Lie group is a connected simply connected real Lie group whose Lie algebra is given, in a suitable basis $X_1, \ldots, X_n$, by nontrivial commutation relations

$$[X_n, X_i] = X_i \qquad (i = 1, \ldots, n-1).$$

Semigroups of the basic affine Lie group and their automorphisms were first analyzed in [2] and, in detail, in [3–6]. Geometrical language was used; and, instead of semigroups, the partial orders generated by them were studied. We should note the article [7], which exerts a great influence on explaining the role of the basic affine Lie group in the theory of Lie semigroups. It was shown that this group occupies a specific position among the ordered Lie groups; later, this was confirmed in [8, 9].

It is important to note that the basic affine Lie group of dimension 4 is the transitive isometry group of the de Sitter stationary universe that was considered by Hoyle and Narlicar as an alternative to the Big Bang theory in cosmology.

Let G be a connected Lie group, and let $P \subset G$ be a semigroup which contains the unity e. Put $P_x \equiv x \cdot P$, by definition. A bijection $f : G \to G$ is said to be a *P-automorphism* if $f(e) = e$ and $f(P_x) = P_{f(x)}$ for each element $x \in G$. Denote the group of all P-automorphisms by $\mathrm{Aut}(P)$.

If N is a one-parameter subgroup, then, given any element x, the set $N_x \equiv x \cdot N$ is called the *line passing through the point x*. A *ray issuing from the point x* is a set of the form $L_x \equiv x \cdot L$, L being a one-parameter semigroup containing the unity.

A *quasiaffine transformation* is defined to be a homeomorphism $f : G \to G$ which maps every line onto a line.

A set of the form $E_x \equiv x \cdot E$, where E is a k-dimensional subgroup, is called a *k-dimensional plane.*

A *cone K_x with vertex x* is the union of rays issuing from the point x.

For a set $A \subset G$, the closure, the interior, and the boundary of A are denoted by $\overline{A}$, $\mathrm{int}\, A$, and ∂A, respectively.

Definition 1. The *quasicontingency* or *q-contingency* of a set $A \subset G$ at a point a is defined to be the cone formed by all limits of rays issuing from a and passing through $x \in A$, where $x \neq a$ and $x \to a$. If a is not a limit point of A, then $\{ a \}$ is called the quasicontingency of A at a. We denote the quasicontingency by $\mathrm{qc}(A, a)$.

It is easy to verify that the q-contingency is a closed cone and $\mathrm{qc}(A, a) = \mathrm{qc}(\overline{A}, a)$.

From now on we will denote the basic affine Lie group of dimension n by H^n.

Let L be a ray with origin e, and let K be a cone with vertex e such that L and K do not lie in

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a common hyperplane and $L \cap K = \{ e \}$. We set

$$L \times K = \bigcup_{x \in A} L(e, x),$$

where $A = \cup_{x \in K} L_x$, and $L(e, x)$ is the ray issuing from e and passing through x, $x \neq e$. The set $L \times K$ is a cone with vertex e; also, if K is a semigroup, then $L \times K$ is also a semigroup.

Let $P \subset H^n$ be a semigroup containing the unity. For $\mathrm{qc}(P, e) \neq L \times K$, the group $\mathrm{Aut}(P)$ has been described [3, Theorem 7]. In the present article we shall give a description of $\mathrm{Aut}(P)$ for the case in which $\mathrm{qc}(P, e) = L \times K$.

Let $M \subset H^n$ be a subset containing the unity; then, by definition, we set $M_x \equiv x \cdot M$, where $x \in H^n$. We obtain the family of subsets, $\mathcal{M} = \{ M_x : x \in H^n \}$. We say that the *family* $\mathcal{M}$ *is preserved by a mapping* $f : H^n \to H^n$ if, for each point $x \in H^n$, $f(M_x) = M_{f(x)}$.

Let E be a k-dimensional subgroup, $1 \leq k \leq n-1$ contained in an Abelian subgroup of H^n. A *k-dimensional horosphere* is defined to be a set $E_x \equiv x \cdot E$, where $x \in H^n$. A horosphere of dimension 1 is called a *horocycle.*

Let L be a ray with origin e. Let E be a hyperplane containing the unity e, and let $L \cap E = \{ e \}$. Assume that either the line N which contains L is a horocycle, or E is a hyperhorosphere. Denote by λ a subset of L which is homeomorphic to the interval $[0, 1]$ so that 0 corresponds to the unity e. Put $\lambda_x \equiv x \cdot \lambda$, where $x \in H^n$; the set λ_x is called an *interval.*

Definition 2. *A displacement* $d_{E\lambda}$ is defined to be a homeomorphism of H^n, $n \geq 2$, onto itself such that:

1) $d_{E\lambda}(e) = e$;

2) for every $x \in H^n$ we have $d_{E\lambda}(\lambda_x) = \lambda_{d_{E\lambda}(x)}$, $d_{E\lambda}(E_x) = E_{d_{E\lambda}(x)}$;

3) $d_{E\lambda}|_E = \mathrm{id}_E$ (i.e., the restriction of $d_{E\lambda}$ to E is the identity mapping).

Let $\partial\lambda = \{ e, a \}$. A set of the form $x \cdot (\lambda \setminus \{ e, a \})$ is called an *open interval with endpoints* x and $x \cdot a$.

Definition 3. A *quasicylinder* $Q(E, \lambda)$ is a subset $M \subset H^n$ meeting the following conditions:

1) M can be represented in the form

$$M = \bigcup_n \left(M_n \cup (M \cap E_n) \right), \tag{1}$$

where $E_n = a^n \cdot E$, n is an integer, and M_n is the union of open intervals with endpoints lying on the planes E_n and E_{n+1} (a priori, some M_n's can be empty);

2) M does not admit a representation (1) with the same E and with an interval $\lambda' \subset L$ such that $\lambda' \neq \lambda$ and $\lambda \subset \lambda'$.

For $\lambda = L$, the displacement d_{EL} and the quasicylinder $Q(E, L)$ are well-defined and exist. If $\lambda \neq L$, then there exist E and λ such that $d_{E\lambda}$ and $Q(E, \lambda)$ are not defined (for example, they do not exist if N is a horocycle; see the proof of Theorem 1).

Let P be a semigroup which contains the unity. We accept the following local Einstein axiom:

(AE) There is a neighborhood of e whose intersection with $\overline{P} \cap \overline{P}^{-1}$ contains only one element, e.

Theorem 1. *Let P be a semigroup, in H^n, containing the unity and satisfying the axiom* (AE). *Assume that* $\mathrm{qc}(P, e) = L \times K$, $L \subset N$ *(N being a horocycle), $K \neq L_1 \times K_1$, and* $\mathrm{int}\,\mathrm{qc}(P, e) \neq \varnothing$. *Then either each continuous P-automorphism is quasiaffine, or $P = Q(E, L)$; i.e.,*

$$P = \left(\bigcup_{x \in U \cap E} L_x \setminus \{ x \} \right) \cup (P \cap E), \tag{2}$$

where E is the hyperplane spanned by K; in this case, every continuous P-automorphism can be written in the form $f_0 \circ d_{EL}$, where f_0 is a quasiaffine transformation, $f_0(E) = E$, and U is some set.

Proof. (A) Let $f : H^n \to H^n$ be a continuous P-automorphism. It follows from [3, Theorem 5] that C, $C = \mathrm{qc}(P, e)$, is a conic semigroup and f is a C-automorphism. Hence, it easily follows that

$$f(E_x) = E_{f(x)}, \qquad f(N_x) = N_{f(x)}. \tag{3}$$

Then $f|_E : E \to E$ preserves the family $\{ K_x : x \in H^n \}$. According to [3, Theorem 1], $f|_E$ is quasiaffine. By [3, Theorem 4], we have

$$f = g \circ \tilde{d}_{EL}, \tag{4}$$

where g is a quasiaffine transformation, $g|_E = f|_E$, $g|_N = \mathrm{id}_N$, and $\tilde{d}_{EL}$ is a displacement. Now we clarify how arbitrary the form of this displacement may be. To this end, we should take into account the fact that f is a P-automorphism.

(B) Let us introduce, in H^n, coordinates $x_1, \dots, x_n$, $x_1 > -1/\sin\theta$, $0 < \theta < \pi$, in which the group operation $x \cdot y$ is given by the rules

$$\begin{aligned}
(x \cdot y)_1 &= [(x_1 \sin\theta + 1)(y_1 \sin\theta + 1) - 1] \cdot (\sin\theta)^{-1}, \\
(x \cdot y)_2 &= (x_1 \sin\theta + 1)(y_1 \cos\theta + y_2) + x_1 \cos\theta + x_2 - (x \cdot y)_1 \cdot \cos\theta, \\
(x \cdot y)_3 &= (x_1 \sin\theta + 1) \cdot y_3 + x_3, \\
&\dots \\
(x \cdot y)_n &= (x_1 \sin\theta + 1) \cdot y_n + x_n;
\end{aligned}$$

the coordinates $x_1, x_3, \dots, x_n$ vary along lines passing through e and lying in E; x_2 varies along N. Also, $x_2 > 0$ on L, and $e = (0, \dots, 0)$.

The coordinate system constructed may be called affine, since in this system any line is given by equations $x_i = a_i \cdot t + b_i$ $(i = 1, \dots, n)$, where $t \in (\delta, +\infty)$, and δ is either a real number or $-\infty$.

As follows from equalities (3), in these coordinates the P-automorphism f takes the form

$$f(x) = (\varphi_1(x_1, x_3, \dots, x_n), \varphi(x_2), \varphi_2(x_1, x_3, \dots, x_n), \dots, \varphi_{n-1}(x_1, x_3, \dots, x_n)).$$

To determine the form of $\varphi(x_2)$, we apply Aleksandrov's method [1; 6.3–6.6] (note that in this article x_2 is denoted by ξ).

As a result, we obtain

$$\varphi(x_2) = x_2 + \vartheta(x_2)$$

(up to a quasiaffine transformation in N of the form $x_2 \to k^{-1}x_2$, with ϑ being a periodic function with periods α's, $\alpha \neq 0$, such that $\vartheta(\alpha) = 0$, and $\alpha \in \partial(P_a \cap N)$, $a \in E$. Thus, we have the following three possibilities:

1. The periods $\alpha \neq 0$ of ϑ are not divisible by any α_0. In this case ϑ is constant; moreover, $\vartheta = 0$, since $\vartheta(\alpha) = 0$. Consequently, f is quasiaffine along N, since $\varphi(x_2) \equiv x_2$.
2. All $\alpha \neq 0$ are multiples of some α_0, where α_0 is the minimal value with this property. Then α_0 is a period of ϑ, with ϑ completely arbitrary in all the other respects.
3. The only admissible value of α is zero. Then φ is any homeomorphism.

In case 1, the displacement $\tilde{d}_{EL}$ is a quasiaffine transformation along N; i.e., f is quasiaffine. Also, P cannot be presented in the form (2), for the semigroup (2) admits a nontrivial displacement d_{EL}.

In case 3, f is an arbitrary homeomorphism along N. This means that P has the form (2). Indeed, the boundary, ∂M, of $M = P_a \cap N$ consists of the single point $\alpha = 0$. Hence, for each point $a \in E$, either P_a contains the whole ray L (or $L \setminus \{ e \}$) lying on N, or $P_a \cap N = \varnothing$. Therefore, the set P intersects every line N_b, $b \in E$, in the ray L_b (or $L_b \setminus \{ b \}$) if $P_a \cap N_b \neq \varnothing$. In this case, P is clearly of the form (2).

Now we show that case 2 cannot occur. Suppose that it does. Then, for each $a \in E$, P_a intersects N in intervals which have lengths divisible by α_0 (with respect to the Euclidean metric that can be defined on N). Let $E_n = a_n \cdot E$, where $E_0 = E$, $a_n \in N$, n is an integer, and the point a_{n+1} is located on N at a distance α_0 from the point a_n. Then f keeps each E_n invariant, for $\vartheta(n \cdot \alpha_0) = 0$. Since ϑ can arbitrarily vary on intervals $(n \cdot \alpha_0, (n+1) \cdot \alpha_0)$, it follows that each line N_b, $b \in E$, intersects P in intervals with endpoints lying on the planes E_n. In the same way, $(H^n \setminus P) \cap N_b$ are intervals with endpoints lying on E_n. However, an analogous statement holds for sets $P_a \cap N$ and $(H^n \setminus P_a) \cap N$, where $a \in E$ is an arbitrary point. Since E is not a horosphere; by moving a into e via a left translation, we find that the sets

$$a^{-1} \cdot (P_a \cap N) = P \cap \left(a^{-1} \cdot N\right)$$

and

$$a^{-1} \cdot [(H^n \setminus P_a) \cap N] = (H^n \setminus P) \cap \left(a^{-1} \cdot N\right)$$

are intervals with endpoints which do not necessarily belong to E_n. Contradiction.

Thus, either f is quasiaffine, and P is not of the form (2), or P is of the form (2), and $f = f_0 \circ d_{EL}$. The proof is complete.

Theorem 2. *Let P be a semigroup, in H^n, which contains the unity and satisfies the axiom* (AE). *Suppose that* $\mathrm{qc}(P, e) = L \times K$, $K \subset E$ *(E being a hyperhorosphere), $K \neq L_1 \times K_1$, and* $\mathrm{int}\, \mathrm{qc}(P, e) \neq \varnothing$. *Then either every continuous P-automorphism is quasiaffine; or P is a quasicylinder, $Q(E, \lambda)$ (the case $\lambda = L$ is not excepted), and $f = f_0 \circ d_E$, f_0 being a quasiaffine transformation.*

Proof. We repeat section (A) of the proof of Theorem 1 to establish that f is of the form (4) and equalities (3) hold.

Let $h = g^{-1} \circ f$, i.e., $h = \tilde{d}_{EL}$. To clarify the form of $h|_N$, we take into account the fact that f is a P-automorphism. Let us introduce, in H^n, coordinates $u_1, \ldots, u_n$, $u_1 > 0$, in which the group operation looks like

$$a \cdot b = (a_1 \cdot b_1, a_1 b_2 + a_2, \ldots, a_1 b_n + a_n), \qquad e = (1, 0, \ldots, 0)$$

[2]; each quasiaffine transformation may be represented by linear expressions; and, moreover, every line is defined by relations of the form $k_i t + \mu_i$ $(i = 1, \ldots, n)$, where t is a parameter. Now, it is easy to verify that h maps P onto the semigroup $h(P)$, i.e., h maps the left-invariant family $\{\, P_x : x \in H^n \,\}$ onto the left-invariant family $\{\, h(P_x) : x \in H^n \,\}$.

It is evident that $h(N_x) = N_{h(X)}$, $h(E_x) = E_{h(X)}$, $h|_E = \mathrm{id}_E$, $h(e) = e$.

Let $M_a = N_a \cap P$, $M'_a = N_a \cap h(P)$, where $a \in E$. Since h is a homeomorphism, the sets M_a and $M'_a = h(M_a)$ are topologically equivalent.

Introduce, on N_a, a left-invariant metric ρ induced by the Lobachevsky metric [2, 3]:

$$ds^2 = u_1^{-2} \cdot \sum_{i=1}^{n} du_i^2.$$

Note that h maps the point of ∂M_a nearest to E (with respect to ρ) into the point of $\partial M'_a$ nearest to E; and, in general, the inequality $\rho(b_1, E) < \rho(b_2, E)$ with $b_i \in \partial M_a$ $(i = 1, 2)$ implies the inequality $\rho(h(b_1), E) < \rho(h(b_2), E)$. Let $\alpha(b_a) = E_{b_a} \cap N$ with $b_a \in \partial M_a$. Then $h(\alpha(b_a)) = \alpha(h(b_a))$. Put $A = \{\, \alpha(b_a) : a \in E \,\}$, $A' = \{\, h(\alpha(b_a)) : a \in E \,\}$, and denote the elements of A and A' by α, $\beta, \ldots$ and α', $\beta', \ldots$, respectively, in such a way that $\alpha' = h(\alpha)$, $\beta' = h(\beta)$, etc. By the above, the inequality $\rho(\alpha, E) < \rho(\beta, E)$ results in $\rho(\alpha', E) < \rho(\beta', E)$.

If A contains only one element, $\alpha = e$, then, obviously, $P = Q(E, L)$, and h is a displacement, d_{EL}. We can show this by analogy to case 3 in section (B) of the proof of Theorem 1. Thus, $f = f_0 \circ d_{EL}$. Let A contain at least two elements. Choose α such that $\alpha \neq e$, $\alpha \in E$, $\alpha = E_b \cap N$, $b \in \partial M_a$, and suppose that b is the point of ∂M_a nearest to E. Then $b^2 \in b \cdot \partial M_a = b \cdot \partial(N_a \cap P) = \partial(N_{ba} \cap P_b)$, and b^2 is the point of $\partial(N_{ba} \cap P_b)$ nearest to E. In general, the point b^2 plays the same role with respect to E_b as that of b with respect to E. Let $\beta = E_{b^2} \cap N$. Since each left translation transforming b into α keeps each horosphere E_x invariant and preserves the family $\{\, N_x : x \in H^n \,\}$, we have

$\rho(e,\beta) = 2\rho(e,\alpha)$. The map h preserves this construction, and, if $\alpha' = h(\alpha)$, $\beta' = h(\beta)$, then, as above, $\rho(e,\beta') = 2\rho(e,\alpha')$, by the left-invariance of the family $\{ h(P_x) : x \in H^n \}$. Repeating the procedure, consider the point $b^3 \in \partial (N_{b^2 a} \cap P_{b^2})$ and find a point $\gamma \in N$ such that $\rho(e,\gamma) = 3\rho(e,\alpha)$ and $h(\gamma) = \gamma' \in N$ with $\rho(e,\gamma') = 3\rho(e,\alpha')$. Continuing in this manner, we generate a sequence of points $\{ \alpha_n \} \subset N$ such that $\alpha_1 = \alpha$, $\alpha_2 = \beta$, $\alpha_3 = \gamma, \dots$, $\rho(e,\alpha_n) = n\rho(e,\alpha)$, $\rho(\alpha_n, \alpha_{n+1}) = \rho(e,\alpha)$, and, moreover, if $\alpha'_n = h(\alpha_n)$, then $\rho(e,\alpha'_n) = n\rho(e,\alpha')$ and $\rho(\alpha'_n, \alpha'_{n+1}) = \rho(e,\alpha')$. If such a sequence can be constructed for any point α, $\alpha \in A$, $\alpha \neq e$, then we say that $h : N \to N$ has α-periodicity.

The following two cases arise: first, among the points of A there exists a point α_0, $\alpha_0 \neq e$, such that each $\rho(e,\alpha)$, $\alpha \in A$, is divisible by $\rho(e,\alpha_0)$; second, there is no such a point.

In the first case, there exist a sequence $\{ (\alpha_0)_n \} \subset N$ and a quasiaffine transformation $F : H \to H$ such that $F|_E = \mathrm{id}_E$, and $F|_N$ is a dilatation. Moreover, $(F^{-1} \circ h)(\alpha_0)_n = (\alpha_0)_n$ (for $n = 1, 2, \dots$) and, on each interval $((\alpha_0)_n, (\alpha_0)_{n+1}) \subset N$, $F^{-1} \circ h$ is an arbitrary homeomorphism. In other words, $f = f_0 \circ d_E$ and $P = Q(E,\lambda)$.

In the second case, we can assert that $h|_N$ is quasiaffine, and so $f|_N$ is quasiaffine. Since, in addition, $f|_E$ is quasiaffine, f is quasiaffine in H^n. Indeed, in this case, there are sequences $\{ \alpha_n \}$, $\{ \beta_n \} \subset N$ (α_1, $\beta_1 \neq e$) such that $\rho(e,\alpha_1)$ is not a multiple of $\rho(e,\beta_1)$. Without loss of generality, we can suppose that $\alpha'_n = h(\alpha_n) = \alpha_n$ (if not, we apply a dilatation along N). Let $\beta'_n = h(\beta_n)$, $\beta'_n \neq \beta_n$ and, for definiteness, assume $\rho(e,\beta_1) < \rho(e,\beta'_1)$ (if not, substitute h^{-1} for h). Suppose that $\rho(\beta_1, \beta'_1) = \varepsilon > 0$. Since $\rho(e,\alpha_1)$ is not a multiple of $\rho(e,\beta_1)$, there are suitable numbers k and m satisfying the inequalities

$$0 < \rho(e,\alpha_m) - \rho(e,\beta_k) < \varepsilon. \tag{5}$$

By α- and β-periodicity of h, we have $\rho(e,\beta'_k) = \rho(e,\beta_k) + k \cdot e$, $\rho(e,\alpha_m) = \rho(e,\alpha'_m)$. Hence $\rho(e,\beta'_k) > \rho(e,\alpha_m)$, contrary to the inequality $\rho(e,\alpha_m) > \rho(e,\beta'_k)$, which follows from (5), since h is a homeomorphism. Thus, $\beta'_n = \beta_n$. Since $\rho(e,\alpha_1)$ and $\rho(e,\beta_1)$ are relatively prime, and h agrees with the identity on $\{ \alpha_n \}$ and $\{ \beta_n \}$, it follows that h is the identity on N; i.e., f is quasiaffine. The proof is complete.

Theorem 3. *Let P be a semigroup, in H^n, containing the unity and satisfying the axiom* (AE). *Assume that* $\mathrm{qc}(P,e) = L \times K$, $K \neq L_1 \times K_1$, $K \subset E$, *the hyperplane E is not a horosphere, and L does not lie on a horocycle. Then every continuous P-automorphism is quasiaffine.*

Proof. The claim follows from [3, Theorem 4(7)].

Remark. Without the requirement $K \neq L_1 \times K_1$, P may be a quasicylinder along several directions. For example, if $P = L \times L_1 \times K_1$, then it is possible that $P = Q(E,L)$ and $P = Q(E_1, L_1)$, where E_1 is the span of L and K_1, and $f = f_0 \circ d_{EL} \circ d_{E_1 L_1}$. This can be proved by using the methods introduced in the proofs of Theorems 1 and 2.

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