Since T_1^* is complete and recursively axiomatizable, by a theorem of Janiczak [5] it is decidable.

The author thanks Yu. L. Ershov for his valuable advice in the completion of this paper.

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MAPPINGS OF AN ORDERED LOBACHEVSKII SPACE

A. K. Guts

UDC 513.812

We consider n-dimensional Lobachevskii space L^n, $n \geqslant 2$ in which there is given an ordering, which is invariant with respect to some simple transitive subgroup T of the group of motions. We pose the problem of the complete description of isotonic homeomorphisms $f: L^n \to L^n$ (i.e., f and f^{-1} are monotonic). In Euclidean space the analogous problem is solved in A. D. Aleksandrov [1].

The results of the paper were announced in [2].

1. Definitions and Notation

(1.1). Geometrically, the introduction of an order in L^n is the assignment to each point $x \in L^n$ of a set $P_x \subset L^n$ satisfying the conditions: 1) $x \in P_x$; 2) if $y \in P_x$, then $P_y \subset P_z$; 3) for $x \neq y$ we have $P_x \neq P_y$. Then writing the relation $y \in P_x$ as $x \leqslant y$ we get a partial ordering in L^n.

The invariance of the order with respect to the group T is understood as follows: If $t \in T$, then $t(P_x) = P_{t(x)}$ for any point $x \in L^n$.

In L^n we fix a point e, and if M is any set in L^n containing the point e, then M_x denotes the set obtained from M with the help of the motion $t \in T$, carrying e into the point x.

A bijective map $f: L^n \to L^n$ of the ordered set L^n is said to be isotonic, if for any point we have $f(P_x) = P_{f(x)}$. It is easy to verify that a bijection f is isotonic if and only if f and f^{-1} are monotonic; i.e., if $x \leqslant y$, then $f(x) \leqslant f(y)$ and $f^{-1}(x) \leqslant f^{-1}(y)$.

(1.2). Let $x_1, \ldots, x_n$ be rectangular Cartesian coordinates in the Euclidean space R^n. By the Poincaré model of Lobachevskii space we mean the half-space $\{x \in \mathbf{R}^n: x_1 > 0\}$, in which the Lobachevskii metric is given by the following differential form:

$$ds^2 = k^2 \frac{\sum_{i=1}^{n} dx_i^2}{x_1^2}, \; k = \text{const} \neq 0.$$

The group T consists of transformations t of the form

$$(x_1, \ldots, x_n) \to (\lambda x_1, \lambda x_2 + \alpha_1, \ldots, \lambda x_n + \alpha_{n-1}),$$

where $\lambda > 0$, $\alpha_1, \ldots, \alpha_{n-1}$ are real numbers, and is a solvable, noncommutative Lie group.

Omsk. Translated from Sibirskii Matematicheskii Zhurnal, Vol. 27, No. 3, pp. 51-67, May-June, 1986. Original article submitted March 21, 1984.

0037-4466/86/2703-0347$12.50 © 1987 Plenum Publishing Corporation

Let ℓ be a line, passing through the point e. We denote by Λ the set of all lines parallel to the line ℓ (in some given direction). Let π be an arbitrary two-dimensional plane, passing through the line ℓ. By the symbol Ψ_π we denote the set of all equidistant curves lying in π and corresponding to the line ℓ. We also introduce the set Φ_π of all horocycle lying on π and orthogonal to ℓ; in addition, if $h \in \Phi_\pi$ is a horocycle, then h, considered as the limit of circles, is characterized by the fact that the centers of the circles cited are taken on the ray $\ell^+ \subset \ell$ which, starting from some point, goes in the direction in which the family of lines Λ is parallel.

We assume further that Λ is represented in the Poincaré model by the coordinate lines x_1. Then to the elements of the sets Φ_π and Ψ_π correspond Euclidean lines (more precisely, intersections of Euclidean lines with $\{x \in \mathbf{R}^n: x_1 > 0\}$).

We denote by Σ the set of all elements gotten from elements of the set $\Sigma_\pi' \equiv \Lambda \cup \Psi_\pi \cup \Phi_\pi$ taken for any plane π, $\ell \subset \pi$ with the help of the group T. I.e., if $\alpha \in \Sigma$ then there exist $t \in T$ and an element $\alpha' \in \Sigma_\pi'$ such that $\alpha = t(\alpha')$. One can write symbolically

$$\Sigma = T\Big(\bigcup_{\pi, l \subset \pi} \Sigma_\pi'\Big).$$

We shall call the elements of the set Σ quasilines (for short, q-lines). In the usual way, from quasilines one can get q-rays, m-dimensional q-planes, etc. In the Poincaré model the intersection of any Euclidean line with the half-space $\{x_1 > 0\}$ is some q-line.

By a q-cone C with vertex at the point e we mean a set which, together with each point x, contains the whole q-ray starting at e and passing through x.

A set $A \subset L^n$ is called q-convex, if along with any two points x and y of it, it contains the whole q-segment with ends x and y.

By $\ell(x, y)$, $x \neq y$ we denote the q-line passing through the points x and y, and by $\ell^+(x, y)$, the q-ray starting from the point x and passing through the point y. We denote the quasisegment with ends x and y by $[x, y]$.

(1.3). We call sets $A, B \subset L^n$ T-parallel, if there exists a motion $t \in T$ such that $t(A) = B$. We say that the family of T-parallel sets $\{M_x: x \in L^n\}$ is preserved under the map $f: L^n \to L^n$ if $f(M_x) = M_{f(x)}$ for any $x \in L^n$.

(1.4). We denote by $|A|$ the object corresponding to the object $A \subset L^n$ under its representation in the Poincaré model. Thus, $|L^n| = \{(x_1, \ldots, x_n) \in \mathbf{R}^n: x_1 > 0\}$. We set $H = \{x_1 = 0\}$.

(1.5). If $A \subset L^n$ then by int A, $\bar{A}$, ∂A we denote respectively the interior, closure, and boundary of the set A.

(1.6). Definition of q-cone $L \times K$. Let L be a q-ray issuing from e, K be a q-cone with vertex e, where L and K do not lie in one q-plane and $L \cap K = \{e\}$. We extent $|L|$, $|K|$ in the natural way in $\mathbf{R}^n$ to a (Euclidean) ray $\tilde{L}$ and cone $\tilde{K}$, respectively. Then by $L \times K$ we shall mean the set such that $|L \times K| = |L^n| \cap (\tilde{L} \times \tilde{K})$. The set $L \times K$ is a q-cone with vertex e, and it can be defined without resorting to the Poincaré model.

In fact,

$$L \times K = \bigcup_{x \in A} l^+(e, x),$$

where $A = \bigcup_{x \in K} L_x$.

(1.7). We shall call a map quasiaffine, if it maps any q-line to a q-line.

Obviously the group T consists of quasiaffine motions.

2. Quasiconical Orders

(2.1). It is easy to see that a q-convex quasicone C defines an invariant order in L^n, i.e., the family of sets $\{C_x: x \in Л^n\}$ generated by it satisfies conditions 1-3 of (1.1) in Sec. 1. We consider three cases:

1) $\bar{C} \neq L \times K$;
2) $C = L \times K$;
3) $C \neq L \times K$, but $\bar{C} = L \times K$.

These three cases exhaust all possibilities which arise in the study of quasiconical orders. The corresponding isotopic homeomorphic maps f are studied in Secs. 3-5. It follows from these sections that in the typical (first) case the map will be quasiaffine. The other two cases are exceptional and the corresponding maps can be rather arbitrary, but nevertheless well described.

(2.2). Isotopic maps f of general orders, i.e., ones which are not quasiconical, will, under specific conditions, preserve certain orders, defined by a q-cone, and consequently in the "typical" case will be quasiaffine [2]. How to do this is shown below in Sec. 6.

(2.3). We note an important fact, frequently used in the course of the proof of theorems. If $\{M_x: x \in E\}$ is some family of sets, where $M \subset E$ is either a q-cone or a union of q-lines, then to study the maps f: $E \to E$ preserving the given family, it makes no difference whether E is the Lobachevskii space L^m, $m \geqslant 1$ or a quasiplane which is not a horosphere.

In fact, it is easy to see this by passing to the Poincaré model.

3. Map f in the Case of a Quasicone $\bar{C} \neq L \times K$

We assume that int $C \neq \emptyset$.

(3.1). THEOREM 1. If an order in L^n, $n \geqslant 2$ is given by a q-cone C, such that ∂C does not contain a q-line and $\bar{C} \neq L \times K$ where L is a q-ray and K is a q-cone of lower dimension, then any homeomorphic isotonic map f is quasiaffine.*

We preface the proof of the theorem with the following lemma.

LEMMA 1. Let σ be an open half-plane, lying in the affine plane τ and $\{\lambda_x^i: x \in \tau\}$ $(i = 1, 2, 3)$ be three different families of parallel lines in τ. If f: $\sigma \to \sigma$ is a homeomorphism such that $f(L_x^i) = L_{f(x)}^i$, where $L_x^i = \sigma \cap \lambda_x^i$, $x \in \sigma$, $i = 1, 2, 3$, then f is affine.

Proof. For a 2-plane σ this result is proved in [1, p. 12]. Let σ be a half-plane. We denote by H the boundary of σ. We extend f to H. Let us assume that L_x^1, L_x^2 are half-lines. We extend L_x^1, L_x^2 to H in the natural way. Let $\{z\} = L_x^1 \cap L_y^2$ and $z \in H$. Then $L_{f(x)}^1 \cap L_{f(y)}^2 \cap H \neq \emptyset$. In fact, let us assume that $L_{f(x)}^1 \cap L_{f(y)}^2 \cap H = \emptyset$. The half-lines $L_{f(x)}^1$ and $L_{f(y)}^2$ bound a domain U such that one can find a point $a \in U$, for which L_a^1, $L_a^2 \subset U$,

$$(L_a^1 \cup L_a^2) \cap L_{f(x)}^1 = \emptyset,\ (L_a^1 \cup L_a^2) \cap L_{f(y)}^2 = \emptyset. \tag{1}$$

Passing to preimages, we note that

$$[f^{-1}(L_a^1) \cup f^{-1}(L_a^2)] \cap [L_x^1 \cup L_y^2] \neq \emptyset,$$

which contradicts (1). Thus, $L_{f(x)}^1 \cap L_{f(y)}^2 = \{z'\}$ and $z' \in H$. Then by definition let $f(z) = z'$. Thus we get a continuous and bijective extension of f to H. Let L_a^1, L_a^2, where $|a| = (0, \ldots, 0)$, be taken on the coordinate axes ξ, η, and the rays $f(L_a^1)$, $f(L_a^2)$ on the coordinate axes ξ', η' in the image. On L_a^3 we take a point b. Through it we draw lines λ_b^1, λ_b^2. Through the points of intersection c_1, c_2 of these lines with λ_a^1, λ_a^2 we draw lines $\lambda_{c_1}^3$, $\lambda_{c_2}^3$, etc. We shall have on σ an integral lattice $\{(n\alpha, m\beta)$: n, m integers$\}$. Since the map f carries parallel lines into parallel ones, the construction is preserved. Consequently, $f(\{(n\alpha, m\beta)\})$ is the integral lattice $\{(n\alpha', m\beta')$: n, m integral$\}$. If $f(\xi, \eta) = (f_1(\xi, \eta), f_2(\xi, \eta))$, then $f_1(\xi, \eta) = f_1(\xi)$, $f_2(\xi, \eta) = f_2(\eta)$, $f_1(n\alpha) = nf_1(\alpha) = n\alpha'$, $f_2(m\beta) = mf_2(\beta) = m\beta'$. This is true for any α, β. Hence, taking $\alpha, \beta \to 0$ and keeping in mind that f is continuous, we get: f is affine.

Remark. The lemma remains valid if we assume that f maps three families of lines (in general position) to three analogous families of lines. The proof of this fact differs inessentially from that given above.

Proof of Theorem 1. Since f is a homeomorphism, we shall assume that $C = \bar{C}$, i.e., C is a closed quasicone.

(a) In our notation $C^- = \{x \in L^n: x \leqslant e\}$, where $\leqslant$ is the order given by the q-cone C. If f is an isotonic bijection, then obviously $f(C_x^-) = C_{f(x)}^-$. We consider the doubly super-

*In [2] we erroneously said "isometric". The latter is valid under additional conditions imposed on C.

ficial q-cone $Q = \partial C \cup \partial C^-$. Let λ be an arbitrary q-line passing through the point e and lying on Q. In the Poincaré model λ is represented by a Euclidean line which can have a point of intersection with the hyperplane $H = \{x \in \mathbf{R}^n: x_1 = 0\}$. Let $x, y \in \lambda$ and $x \neq y$. We shall say that the q-ray $\ell^+(x, y)$ is free, if its representation in the Poincaré model does not intersect the hyperplane H.

Let now y be a point on the boundary ∂Q_x such that the q-ray $\ell^+(x, y)$ is free. Let

$$M_{xy} = \bigcup C_z^- \{z \in l^+(x, y)\}, \quad \text{if} \quad l^+(x, y) \subset \partial C_x,$$

and

$$M_{xy} = \bigcup C_z \{z \in l^+(x, y)\}, \quad \text{if} \quad l^+(x, y) \subset \partial C_x^-.$$

If $\bar{M}_{xy}$ happens to be a q-half-space, then $\tau = \partial M_{xy}$ is the q-tangent q-plane of the quasicone C_y^- at the point x and at the same time (by symmetry) q-tangent to the q-quasicone C_x at the point y. Conversely, if at y the quasicone C_x has q-tangent q-plane τ, then $\tau = \partial M_{xy}$ when the ray $\ell^+(x, y)$ is free, and $\tau = \partial M_{xu}$, where $u \in l(x, y) \setminus l^+(x, y)$, when $\ell^+(x, y)$ is not a free ray.

In general $\bar{M}_{xy}$ is represented in the Poincaré model by a convex cone which is a dihedral angle containing the line passing through the points x and y. Let R_{xy} be a q-plane of highest dimension passing through x and lying in $\bar{M}_{xy}$.

If M_{uv} is defined, then as is easy to verify, $\bar{M}_{uv} = \bar{M}_{xy}$ if and only if $u, v \in R_{xy}$. Consequently, R_{xy} is the set of all u for which there exist points v such that $M_{uv} = \bar{M}_{xy}$.

Consequently, the sets $\bar{M}_{xy}$ and R_{xy} are defined only in terms of the order and topology. Hence, they are preserved under a continuous map f such that $f(\bar{M}_{xy})$ and $f(R_{xy})$ have the same meaning.

As we explained above, the quasicone C_x has a q-tangent q-plane τ at the point y, if and only if there exists a point u, possibly equal to y, such that $\tau = \partial M_{xu}$. Hence $R_{xu} = \partial M_{xu}$. The homeomorphism f preserves this equality, which is the condition defining the q-tangent q-plane. Hence, to q-tangent q-planes of quasicones C_x correspond q-tangent q-planes of quasicones $C_{f(x)}$ and conversely.

(b) q-Tangent T-parallel q-planes τ_1, τ_2, $\tau_2 = t(\tau_1)$, where $t \in T$, are mapped to q-tangent T-parallel q-planes τ_1' and τ_2' respectively, for which there exists a $t' \in T$ such that $\tau_2' = t'(\tau_1')$ (in the Poincaré model, the q-planes τ_1, τ_2 and τ_1', τ_2' are pictured as pairs of parallel planes $|\tau_1|$, $|\tau_2|$ and $|\tau_1'|$, $|\tau_2'|$). The converse is also true.

First let us assume that $\tau_2' = t'(\tau_1')$. We show that $\tau_2 = t(\tau_1)$.

In fact, the q-planes τ_1', τ_2' do not intersect. Hence their preimages $\tau_1 = f^{-1}(\tau_1')$, $\tau_2 = f^{-1}(\tau_2')$ do not intersect. If τ_1 is pictured in the model plane by a parallel of the hyperplane H, then obviously so is τ_2, i.e., $\tau_2 = t(\tau_1)$ for some element $t \in T$. In general, if $\tau_2 \neq t(\tau_1)$ then the q-planes τ_1 and τ_2 bound a q-convex closed domain U such that if $x \in U$ then either $C_x \subset U$ or $C_x^- \subset U$. For definiteness let the first inclusion hold. Since $\tau_1 \cap \tau_2 = \varnothing$, one can find points $u \in \tau_1$ and $v \in \tau_2$ such that $C_u \cap \tau_2 = \varnothing$ and $C_v \cap \tau_1 = \varnothing$. Then $C_{f(u)} \cap \tau_2' = \varnothing$ and $C_{f(v)} \cap \tau_1' = \varnothing$. But since $\tau_2' = t'(\tau_1')$, at least one of these equations is false. We have found a contradiction. Consequently, $\tau_2' = t'(\tau_1')$ implies $\tau_2 = t(\tau_1)$.

Now let $\tau_2 = t(\tau_1)$. We set $\tau_1' = f(\tau_1)$, $\tau_2' = f(\tau_2)$. To be definite, we assume that τ_2 is q-tangent to C_x and τ_1 to C_y. Then τ_1 is q-tangent to $C_{f(y)}$ and there exists a q-plane $t'(\tau_1')$ q-tangent to $C_{f(x)}$. As was established above, to the quasiplane $t'(\tau_1')$ corresponds the q-plane τ_2. But then $f(\tau_2) = t'(\tau_1') = \tau_2'$.

Thus, the conditions $\tau_2 = t(\tau_1)$, $\tau_2' = t'(\tau_1')$ imply one another under the homeomorphism f.

(c) The q-tangent q-planes of the quasiconic C bound it. Hence one can take n q-tangent q-planes τ_i $(i = 1, \ldots, n)$ bounding the n-faces of the quasiangle V. Since under the map f the q-tangent q-planes go into q-tangents, and T-parallels into T-parallels, the edges of the quasiangles V_x too go into edges of the quasiangles $V_{f(x)}$ which are compatible with the help of the group T. We take any edge L of the quasiangle V. The quasicone C has q-tangent q-planes which differ from τ_i since otherwise $C = V$, i.e., $|C|$ would be a Cartesian product. All such q-tangent q-planes cannot pass through the edge L, because if they did, one would have $C = L \times K$. Hence there is a q-tangent q-plane τ not passing through the edge L and differing from the q-plane τ_i opposite to it. Hence, besides L there is at least one more edge N, not contained in τ. The quasiplane σ spanned by L and N intersects τ in a q-line $S = \sigma \cap \tau$. Thus, we have on σ three families of q-lines, T-parallel respectively to L, N, and S.

Under the map f the q-lines which are T-parallel to L and N go into T-parallels. Hence the q-planes σ_x go into two-dimensional q-planes $\tilde{\sigma}_{f(x)}$, the q-tangent q-planes τ_x go into q-tangent q-planes $\tilde{\tau}_{f(x)}$, so that the q-lines $S_x = \sigma_x \cap \tau_x$ go into the q-lines $\tilde{S}_{f(x)} = \tilde{\sigma}_{f(x)} \cap \tilde{\tau}_{f(x)}$.

If one now makes use of the Poincaré model, then $|\sigma_x|$, $|\tilde{\sigma}_x|$ are affine half-planes (or planes), and $\{|L_x|, |N_x|, |S_x|\}$ are three families of parallel half-lines (lines), mapped onto the corresponding three families of parallel half-lines (lines) $\{|\tilde{L}_{f(x)}|, |\tilde{N}_{f(x)}|, |\tilde{S}_{f(x)}|\}$.

It is easy to see that if $|\sigma|$ is a half-plane, then $|f(\sigma)| = |\tilde{\sigma}|$ is also a half-plane. Applying Lemm 1, we see that $|f|$ is affine on $|\sigma|$ in the Poincaré model, and consequently, the original map f is quasiaffine on the q-plane σ.

The map $|f|$ maps lines lying in $|\sigma|$ and parallel to H to lines parallel to H. In our arguments the edge L was chosen arbitrarily. In all there are n edges of the quasiangle V. Hence each of them lies on some two-dimensional q-plane on which f is quasiaffine.

Among these q-planes one can take $(n-1)$ q-planes $\sigma_1, \ldots, \sigma_{n-1}$ such that the lines passing through the point e, lying in $|\sigma_i|$ and parallel to the hyperplane H, are in general position. Let them correspond to the q-lines $\lambda_1, \ldots, \lambda_{n-1}$. By what was said above, $|f|(|\lambda_{ix}|) = |\tilde{\lambda}_{if(x)}|$, where $|\tilde{\lambda}_i|$ are lines parallel to the hyperplane H. Without loss of generality we assume that the edge $|L|$ is not parallel to H. We take the lines $|L|$, $|\lambda_i|$ $(i = 1, \ldots, n-1)$ as axes of an affine coordinate system in the half-space $|L^n| = \{x_1 > 0\}$. Since $|f|$ is affine on these axes, $|f|$ is affine in $|L^n|$. But then f is quasiaffine in L^n. Theorem 1 is proved.

THEOREM 1'. The assertion of Theorem 1 remains valid if we assume that f(c) is a quasicone, and $f(C_x)$ is gotten from f(C) with the help of the group T for any $x \in L^n$.

The proof of Theorem 1' does not differ from the proof of Theorem 1.

4. The Case C = L × K

In what follows we shall need

(4.4). LEMMA 2. If $f: L^n \to L^n$ $(n \geqslant 2)$ is a homeomorphism, preserving the family of q-lines $\{N_{ix}: x \in L^n\}$ $(i = 1, \ldots, n)$ (i.e., $f(N_{ix}) = N_{if(x)}$ for any $x \in L^n$ $(i = 1, \ldots, n)$), which is the identity on q-lines $N_1, \ldots, N_n$ passing through the point e and in general position, then f is the identity on L^n.

Proof. If among the q-lines $N_1, \ldots, N_n$ there exist $(n-1)$ lines which are a horocycle, then the assertion of the lemma is obvious (cf. the end of the proof of Theorem 1). Hence we assume that there are not such q-lines. We denote by Q the q-hyperplane spanned by $N_1, \ldots, N_{n-1}$. Further, in the natural way we complete $|Q|$ and $|N_n|$ to a hyperplane and a line in R^n, but we leave the notation unchanged, i.e., $|Q|$ and $|N_n|$. By hypothesis Q is not a hyperhorosphere, i.e., $|Q|$ is not parallel to H. If N_n is a horocycle, then we set $U = |L^n|$. Now we define the domain U when N_n is not a horocycle. Let $|\sigma_1|$ be the hyperplane obtained by taking the union of all lines $|N_{nx}|$ such that $|N_{nx}| \cap H \cap |Q| \neq \varnothing$, and $|\sigma_2|$ coincides with some $|Q_y|$ such that $|Q_y| \cap H \cap |N_n| \neq \varnothing$. We denote by $U \subset |L^n|$ the closed domain bounded by the hyperplanes $|\sigma_1|$, $|\sigma_2|$.

The rest of the proof is by induction on the dimension n.

A. n = 2. Let $x \in \operatorname{int} U$, so $N_{1x} \cap N_2 \neq \varnothing$ and $N_{2x} \cap N_1 \neq \varnothing$; hence since f is the identity on N_1, N_2 it follows that $f(x) = x$, i.e., f is the identity on int U, and hence also on U.

We take $x \notin U$. Then either $N_{1x} \cap N_2 \neq \varnothing$, or $N_{2x} \cap N_1 \neq \varnothing$. To be definite we take the first; the second case can be considered analogously. Since f is the identity on N_2 f(x) will lie on N_{1x} by virtue of the fact that $f(N_{1x}) = N_{1x}$. Let $\{a\} = |N_{2x}| \cap H$ and $b \in N_2$ be such that $|N_{1b}| \cap H = \{a\}$. There exists a sequence $\{b_m\} \subset N_2$ for which $b_m \xrightarrow[m\to\infty]{} b$ and

$N_{1b_m} \cap N_{2x} \neq \varnothing$ $(m = 1, 2, \ldots)$. Since $f(b_m) = b_m$, $f(b) = b$, one has $f(N_{1b_m}) = N_{1b_m}$, $f(N_{1b}) = N_{1b}$, so $f(N_{2x}) \cap N_{1b_m} \neq \varnothing$, $m = 1, 2, \ldots$, $f(N_{2x}) \cap N_{1b} = \varnothing$. It follows from this that $f(N_{2x}) = N_{2x}$, i.e., $\{x\} = N_{1x} \cap N_{2x}$ remains fixed under map f. Thus the lemma is proved if n = 2.

B. Let the lemma be valid for dimension $k \leqslant n - 1$ and suppose we are in the case k = n. Then in the half-hyperplane $|Q|$ there will be a family of preserved q-lines $\{N_{ix}: x \in Q\}$ $(i = 1, \ldots, n-1)$, while $f(Q) = Q$. Since the consideration of L^{n-1} reduces (cf. point (2.3)) to the study of the half-space $|L^{n-1}|$, one has that $|Q|$ does not differ in this from $|L^{n-1}|$ and one can use the induction hypothesis, considering f to be the identity on Q. But then it is easy to conclude that $|f|$ is the identity on $U \subset |L^n|$. Now let $x \notin U$. We take the q-plane S spanned by N_{1x}, N_{nx}. Since S goes, under the map f, into the q-plane S' spanned by $N_{1f(x)}$, $N_{nf(x)}$ and $S \cap Q \neq \varnothing$, and f is the identity on Q, one has $f(S) = S$, i.e., $S' = S$. On S, f preserves the two families of q-lines $\{N_{1x}: x \in S\}$, $\{N_{nx}: x \in S\}$. Since upon reducing consideration from S to $|S|$, $|S|$ in no way differs in the present situation from $|L^2|$, one can use the assertion of point A, i.e., assume it proved that $|f|$ is the identity on $|S|$ and hence $f(x) = x$, i.e., f is the identity on L^n. The lemma is proved.

(4.2). Definition of displacement. Let L be a q-ray issuing from e, and E be a q-hyperplane containing the point e, and $L \cap E = \{e\}$. Let us assume that either N is a horocycle, where N is a q-line containing L, or E is a hyperhorosphere. Then we have

Definition 1. A displacement of the first kind d_{EL} is a homeomorphism of L^n, $n \geqslant 2$ to itself such that

1) d_{EL} is an (arbitrary) homeomorphism on N;

2) for any point $x \in L^n$ we have $d_{EL}(L_x) = L_{d_{EL}(x)}$, $d_{EL}(E_x) = E_{d_{EL}(x)}$;

3) $d_{EL}|_E$ is a motion from T (i.e., under the condition $d_{EL}(e) = e$ the displacement d_{EL} is the identity on E).

It follows from the definition that d_{EL} maps any q-line, T-parallel to the q-plane E, to another such. Now let L_1, L_2 be two different q-rays, issuing from the point e, N_1, N_2 be q-lines containing them respectively, which are not horocycle, E_1 be a q-hyperplane passing through N_2 and $E_1 \cap L_1 = \{e\}$. Then we have

Definition 2. A displacement of the second kind $d_{E_1L_1L_2}$ is a homeomorphism of L^n, $n \geqslant 2$ to itself such that

1) $d_{E_1L_1L_2}$ is an (arbitrary) homeomorphism on N_1;

2) for any $x \in L^n$ we have

$$d_{E_1L_1L_2}(L_{jx}) = L_{j d_{E_1L_1L_2}(x)} \ (j = 1, 2), \ d_{E_1L_1L_2}(E_{1x}) = E_{1 d_{E_1L_1L_2}(x)};$$

3) $d_{E_1L_1L_2}|_{U_1}$ is a motion from T, where

$$U_1 = \overline{E_1 \cap \bigcup_{x \in E_1^0} N_{1x}}, \ E_1^0 = \partial \Big(\bigcup_{E_{1x} \cap N_1 \neq \varnothing} E_{1x} \Big)$$

(i.e., under the condition $d_{E_1L_1L_2}(e) = e$ the motion $d_{E_1L_1L_2}$ is the identity on U_1 or $d_{E_1L_1L_2}|_{U_2} = \mathrm{id}_{U_1}$).

LEMMA 3. If d is a displacement of the second kind $d_{E_1L_1L_2}$ such that $d(e) = e$, then

1) $d(E_1) = E_1$, $d(N_1) = N_1$, $d(E_1^0) = E_1^0$, $d(U_1) = U_1$, $d(S_1) = S_1$, where $S_1 = \partial \Big(\bigcup_{N_{1x} \cap E_1^0 = \varnothing} N_{1x} \Big)$;

2) d preserves the families $\{E_{1x}^0\}$, $\{U_{1x}\}$, $\{S_{1x}\}$ and the family of $(n-2)$-dimensional q-planes $\{\pi_{1x}\}$, where $\pi_{1x} = S_{1x} \cap E_{1x}$;

3) d preserves the family of q-lines $\{N_x\}$ provided $N_x \subset \pi_{1x}$, i.e., $d(N_x) = N_{d(x)}$ for any point $x \in L^n$.

Proof. Assertions 1 and 2 are obvious. We prove 3. We take a point $x \in L^n$ arbitrarily. If $x \in U_1$, since $N_x \subset \pi_{1x} \subset U_1$ and $d|_{U_1} = \mathrm{id}_{U_1}$, one has $d(N_x) = N_x$. If $x \in E_1 \setminus U_1$, then $N_x = \pi_{1x} \cap \sigma_x$, where σ_x is the two-dimensional q-plane spanned by L_{2x} and N_x. But $\sigma_x \cap \pi_1$ is a q-line, T-parallel to N_x (and $|\sigma_x \cap \pi_1|$ is parallel to H), preserved under the displacement d, since it lies in U_1. Since d preserves the family $\{L_{2z}: z \in L^n\}$ by Definition

2, it preserves the q-plane σ_x, or more precisely $d(\sigma_x) = \sigma_x$. But then d preserves N_x, because $d(N_x) = d(\pi_{1x}) \cap d(\sigma_x) = \pi_{1d(x)} \cap \sigma_x$ is a q-line, T-parallel to the q-line N_x, i.e., the line $N_{d(x)}$.

Now let $x \notin E_1$. Then $N_x = \Pi_{1x} \cap \tau_x$, where τ_x is the two-dimensional q-plane spanned by N_{1x} and N_x. Either $N_{1x} \cap E_1 \neq \varnothing$, or $N_{1x} \cap E_1 = \varnothing$. If $N_{1x} \cap E_1 \neq \varnothing$, then $\tau_x \cap E_1$ is a q-line, T-parallel to N_x, lying in E_1, and consequently, preserved under the displacement d, as was proved above. But then it follows from the preservation of $\tau_x \cap E_1$ and the preservation of the family $\{N_{1z}: z \in \Pi^n\}$ under the displacement d that $d(\tau_x)$ is the two-dimensional q-plane, T-parallel to τ_x. But then $d(N_x) = d(\pi_{1x} \cap \tau_x) = \pi_{1d(x)} \cap d(\tau_x)$ is a q-line, T-parallel to N_x, or $d(N_x) = N_{d(x)}$. Finally, if $N_{1x} \cap E_1 = \varnothing$, we have

$$N_x = \Big(\bigcup_{y \in N_z} N_{2y}\Big) \cap \pi_{1x},$$

where N_z is such that $N_{1z} \cap E_1 \neq \varnothing$ (here one uses the fact that N_2 is not a horocycle). Since as was just proved $d(N_z) = N_{d(z)}$ and also the families $\{N_{2y}: y \in L^n\}$ and $\{\pi_{1y}: y \in L^n\}$ are preserved, we conclude directly that $d(N_x) = N_{d(x)}$. Lemma 3 is proved.

Remark. Assertions 2 and 3 of Lemma 3 remain valid without the condition $d(e) = e$.

(4.3). Let $f: L^2 \to L^2$ be a homeomorphism such that $f(N_{ix}) = N_{if(x)}$ $(i = 1, 2)$, $f(e) = e$, where N_1, N_2 are two different q-lines, which are not horocycles and pass through the point e.

Each N_i is the union of two q-rays n_i and $L_i = (N_i \backslash n_i) \cup \{e\}$ of which L_i is a free ray (cf. the beginning of the proof of Theorem 1). If $x \in n_i$, then $f(x) \in n_i$. In fact, to be definite let $x \in n_1$. If $|N_{2x}| \cap |N_{1x}| \cap H$ is a point ($|N_{2x}|$, $|N_{1z}|$ extend naturally to lines in $\mathbf{R}^2$), then $|N_{2f(x)}| \cap |N_{1f(z)}| \cap H$ is also a point, as is shown in Lemma 1. From this it follows that if $f(x) \in L_1$, then $N_{1f(z)} \cap N_2 = \varnothing$. The latter contradicts the fact that $N_{1z} \cap N_2 \neq \varnothing$. Hence $f(x) \in n_1$.

Knowing how f acts on n_1, it is easy to determine the action of f on n_2. In this sense the actions of f on N_1, N_2 are dependent. In fact, if $x \in n_1$, then f: $|N_{2x}| \cap H \to |N_{2f(x)}| \cap H$. Hence if $y \in n_2$ is such that $|N_{1y}| \cap H = |N_{2x}| \cap H$, then $|N_{1f(y)}| \cap H = |N_{2f(x)}| \cap H$. In other words, the latter equation determines the location of the point $f(y)$ on n_2 which depends on the point $f(x)$.

(4.4). THEOREM 2. Let $C = L_1 \times L_2$, where L_1, L_2 are different q-rays, issuing from the point e, let the order in L^2 be such that ∂C does not contain a q-line. Then any C-isotonic homeomorphism f can be represented in one of the two forms:

$$f = f_0 \circ d_{N_1 L_2 L_1} \circ d_{N_2 L_1 L_2} \tag{2}$$

or

$$f = f_0 \circ d_{N_2 L_1} \circ d_{N_1 L_2}, \tag{3}$$

where f_0 is a quasiaffine transformation. In addition, the displacements in (2) and (3) commute. In (2) the displacements of the second kind are not independent (cf. (4.3)), and the displacement in (3) is completely arbitrary.

Proof. Since f is a homeomorphism it has the property that the edges L_{1x}, L_{2x} of the q-cones C_x map to edges. Obviously there exists a q-affine bijection f_0: $L^2 \to L^2$ such that $f_0^{-1}(f(L_{ix})) = L_{if_0^{-1}(f(x))}$ $(i = 1, 2)$, $(f_0^{-1} \circ f)(e) = e$. Hence if $g = f_0^{-1} \circ f$ then $g(e) = e$, $g(C_x) = C_{g(x)}$ and $g(L_{ix}) = L_{ig(x)}$ $(i = 1, 2)$ for any point $x \in L^2$. It remains to show that g can be represented as a composition of two displacements of the same kind.

A. First let us assume that N_1 is a horocycle. We take a displacement $d_1 = d_{N_2 L_1}$ such that

$$d_1|_{N_1} = g|_{N_1},\ d_1(e) = e.$$

Then $h_1 = d_1^{-1} \circ g$ has the following property:

$$h_1|_{N_1} = \mathrm{id}_{N_1}, \tag{4}$$

i.e., is the identity on N_1. Since d_1 preserves the families of lines $\{N_{1x}: x \in L^2\}$, $\{N_{2x}: x \in L^2\}$, one has that h_1 will be a C-isotonic homeomorphism. Now let $d_2 = d_{N_1 L_2}$ be a displacement such that

$$d_2|_{N_2} = h_1|_{N_2},\ d_2(e) = e.$$

Then if $h_2 = d_2^{-1} \circ h_1$ then

$$h_2|_{N_2} = \mathrm{id}_{N_2}. \tag{5}$$

But d_2 preserves the families $\{N_{1x}\}$ and $\{N_{2x}\}$ and, moreover, is the identity on N_1. From this is follows that h_2 preserves the families $\{N_{1x}\}$, $\{N_{2x}\}$ and by (4),

$$h_2|_{N_1} = \mathrm{id}_{N_1}. \tag{6}$$

Having (5) and (6) in mind, we can apply Lemma 2 to h_2. We get that $h_2 = \mathrm{id}_{L^2}$, i.e., $g = d_1 \circ d_2$ or $f = f_0 \circ d_1 \circ d_2$. That d_1, d_2 commute is obvious.

B. Now let the q-lines N_1, N_2 not be horocycles. Then g has a representation in terms of displacements of the second kind.

The actions of the displacements of the first kind $d_{N_1L_2}$, $d_{N_2L_1}$ on N_1, N_2 respectively are comletely independent, which cannot be said of g and the displacements of the second kind $d_{N_1L_2L_1}$, $d_{N_2L_1L_2}$. This is discussed in point (4.3). Let $d_1 = d_{N_1L_2L_1}$ be a displacement such that

$$d_1|_{N_2} = g|_{N_2},\ d_1(e) = e.$$

Then $h_1 = d_1^{-1} \circ g$ has the following properties:

$$h_1|_{N_2} = \mathrm{id}_{N_2} \tag{7}$$

and it preserves the families of lines $\{N_{1x}\}$, $\{N_{2x}\}$, i.e., is a C-isotonic homeomorphism. By virtue of the dependence of g and d_1 on n_1, n_2 mentioned above, we get that $d_1|_{n_1} = g|_{n_1}$ or

$$h_1|_{n_1} = \mathrm{id}_{n_1}. \tag{8}$$

Now let $d_2 = d_{N_2L_1L_2}$ be such that

$$d_2|_{N_1} = h_1|_{N_1},\ d_2(e) = e. \tag{9}$$

Then it follows from (8) and (9) that

$$d_2|_{n_1} = \mathrm{id}_{n_1},\ d_2|_{n_2} = \mathrm{id}_{n_2}. \tag{10}$$

The latter is valid again by virtue of the dependence of the action of d_2 on n_1 and n_2. Since by Definition 2 d_2 is the identity on $U_1 = N_2 \backslash n_2$, it follows from (10) that

$$d_2|_{N_2} = \mathrm{id}_{N_2}. \tag{11}$$

As a result, the homeomorphism $h_2 = d_2^{-1} \circ h_1$ has the following properties: it preserves the families $\{N_{1x}\}$, $\{N_{2x}\}$ and by (7), (9), and (11),

$$h_2|_{N_1} = \mathrm{id}_{N_1},\ h_2|_{N_2} = \mathrm{id}_{N_2}.$$

But then by Lemma 2 we get that h_2 is the identity on L^2 so $f = f_0 \circ d_1 \circ d_2$. The commutation of d_1, d_2 is obvious. Theorem 2 is proved.

(4.4). Suppose given in L^n, $n \geqslant 2$, an order $C = L_1 \times \ldots \times L_n$, where int $C \neq \varnothing$. Here L_i ($i = 1, \ldots, n$) are q-rays issuing from the point e. As before, we denote by N_i the q-line containing the q-ray L_i, and by E_i the q-hyperplane spanned by $N_1, \ldots, N_{i-1}, N_{i+1}, \ldots, N_n$.

Lemma 4. Let $f\colon L^n \to L^n$, $n \geqslant 3$, be a homeomorphism such that $f(N_{ix}) = N_{if(x)}$ ($i = 1, \ldots, n$) and the q-lines $N_1, \ldots, N_n$ are in general position and are not horocycles. Then f is quasiaffine.

Proof. For $n = 3$ we consider, on the q-plane E_1, the three families of lines $\{N_{2x}\}$, $\{N_{3x}\}$, and $\{E_1 \cap S_{1x}\}$. By hypothesis the map f is such that $f(e) = e$, and without loss of generality we can assume it has the property $f(E_1) = E_1$ and preserves the cited families of q-lines. Then by Lemma 1, f is quasiaffine on E_1 and hence on N_2 and N_3. From the symmetry of the q-lines N_1, N_2, N_3 in our investigation, we conclude that f is also q-affine on N_1. One can find a q-affine transformation f_0 such that $f_0|_{N_i} = f|_{N_i}$ ($i = 1, 2, 3$). Hence if $g = f_0^{-1} \circ f$, then $g|_{N_i} = \mathrm{id}_{N_i}$ ($i = 1, 2, 3$). By Lemma 2, in this case g is the identity on L^3, i.e., $f = f_0$. The case $n = 3$ is proved.

In general it is easy to see that f is q-affine on each q-line N_i, because each such line can be included in the three-dimensional q-plane σ spanned by the three q-lines N_i, N_j, N_k. Since the study of $|\sigma|$ does not differ (cf. (2.3)), in the direction of interest to us, from the study of $|L^3|$ we see, just as above, that f is q-affine on σ and hence also on N_i. From here on we argue just as in the case n = 3. Lemma 4 is proved.

THEOREM 3. Let $f\colon L^n \to L^n$ $(n \geqslant 3)$ be a C-isotonic homeomorphism, where $C = L_1 \times \ldots \times L_n$ and ∂C does not contain a q-line; then

1) if all N_i (i = 1, ..., n) are not horocycles, then f is quasiaffine;

2) if only N_1, N_2 are not horocycles, then

$$f = f_0 \circ d_{E_1L_1L_2} \circ d_{E_2L_2L_1} \circ d_{E_3L_3} \circ \ldots \circ d_{E_nL_n}; \tag{12}$$

3) if N_1, ..., N_k $(k \geq 3)$ are not horocycles, and N_{k+1}, ..., N_n are horocycles, then

$$f = f_0 \circ d_{E_{k+1}L_{k+1}} \circ \ldots \circ d_{E_nL_n}; \tag{13}$$

4) if only N_1 is not a horocycle, then

$$f = f_0 \circ d_{E_1L_1} \circ \ldots \circ d_{E_nL_n}. \tag{14}$$

Here f_0 is a quasiaffine transformation, all the displacements in (12)-(14) commute. In addition, any displacements of the first kind are admissible, and for the displacements of the second kind one should consider point (4.3).

Proof. The map f maps each family $\{N_{ix}\}$ to some family $\{N_{jx}\}$. In fact, each N_{ix} is the intersection of the q-planes E_{1x}, ..., $E_{i-1,\,x}$, $E_{i+1,\,x}$, ..., E_{nx}, which are q-tangent to C_x. Since f maps q-tangent q-planes to q-tangent q-planes (cf. the proof of Theorem 1), we conclude directly that $f(N_{ix}) = N_{jf(x)}$ for any point $x \in L^n$.

One can find a quasiaffine bijection $f_0\colon L^n \to L^n$ such that if $g = f_0^{-1} \circ f$, then

$$g(N_{ix}) = N_{ig(x)} \quad (i = 1, \ldots, n), \quad g(e) = e.$$

Case 1. According to Lemma 4, the map g is quasiaffine, so f is also.

Case 2. Let $d_1 = d_{E_1L_1L_2}$ be a displacement having the property that

$$d_1|_{N_1} = g|_{N_1}.$$

Then $h_1 = d_1^{-1} \circ g$ is the identity on N_1, and by the dependence of the action of g and d_1 on N_1, N_2 we get that

$$h_1|_{n_2} = \mathrm{id}_{n_2}, \tag{15}$$

where we have used the notation of the proof of Theorem 2. In accord with Lemma 3 and Definition 2 the displacement d_1 preserves the families of q-lines $\{N_{ix}\}$ (i = 1, ..., n). Hence h_1 also preserves them, i.e., h_1 is a C-isotonic homeomorphism. Let $d_2 = d_{E_2L_2L_1}$ be a displacement such that

$$d_2|_{N_2} = h_1|_{N_2}. \tag{16}$$

Then by Definition 2 and Lemma 3, the displacement d_2 is also C-isotonic and preserves the families $\{N_{ix}\}$ (i = 1, ..., n) and in addition, by the dependence of the action of d_2 on N_1, N_2 we get from (15) and (16) that

$$d_2|_{n_1} = \mathrm{id}_{n_1}.$$

Since d_2 is the identity on $U_2 \supset N_1 \backslash n_1$ one has that d_2 is the identity on N_1. Hence if $h_2 = d_2^{-1} \circ h_1$, one has

$$h_2|_{N_2} = \mathrm{id}_{N_2}, \; h_2|_{N_1} = \mathrm{id}_{N_1}.$$

Moreover, h_2 preserves the families $\{N_{ix}\}$ (i = 1, ..., n). We take a displacement $d_3 = d_{E_3L_3}$ such that

$$d_3|_{N_3} = h_2|_{N_3}.$$

Then it follows from Definition 1 that if $h_3 = d_3^{-1} \circ h_2$, then h_3 preserves the families $\{N_{ix}\}$ (i = 1, ..., n) and

$$h_3|_{N_j} = \mathrm{id}_{N_j} \quad (j = 1, 2, 3).$$

Continuing this process, at the (n − 2)nd step we take a displacement $d_n = d_{E_nL_n}$ such that

$$d_n|_{N_n} = h_{n-1}|_{N_n}.$$

And then for $h_n = d_n^{-1} \circ h_{n-1}$ on the basis of Definition 1,

$$h_n|_{N_j} = \mathrm{id}_{N_j} \quad (j = 1, 2, \ldots, n).$$

By Lemma 2 we get $h_n = \mathrm{id}_{L^n}$, i.e., (12) holds for f.

Case 3. Let σ be the quasiplane spanned by $N_1, \ldots, N_k$. Then $g(\sigma) = \sigma$ since g is a homeomorphism, and in view of Lemma 4, the map g is quasiaffine on σ. Now let f_1 be a quasiaffine map which coincides with g on σ and preserves the families $\{N_{ix}\}$ $(i = 1, \ldots, n)$. Then $h = f_1^{-1} \circ g$ is the identity on σ and preserves the families $\{N_{ix}\}$ $(i = 1, \ldots, n)$. Let $d_1 = d_{E_{k+1}L_{k+1}}$ be a displacement such that

$$d_1|_{N_{k+1}} = h|_{N_{k+1}}.$$

Then if $h_1 = d_1^{-1} \circ h$ then h_1 preserves the families $\{N_{ix}\}$ $(i = 1, \ldots, n)$ and

$$h_1|_{N_j} = \mathrm{id}_{N_j} \ (j = 1, \ldots, k+1)$$

by Definition 1. Then we take $d_2 = d_{E_{k+2}L_{k+2}}$ so that

$$d_2|_{N_{k+2}} = h_1|_{N_{k+2}},$$

and we consider $h_2 = d_2^{-1} \circ h_1$. The map h_2 preserves the families $\{N_{ix}\}$ $(i = 1, \ldots, n)$ and

$$h_2|_{N_j} = \mathrm{id}_{N_j} \ (j = 1, \ldots, k+2)$$

by Definition 1. Finally, at the $(n - k)$th step, we introduce the displacement $d_{n-k} = d_{E_nL_n}$ such that

$$d_{n-k}|_{N_n} = h_{n-k-1}|_{N_n}.$$

Then if $h_{n-k} = d_{n-k}^{-1} \circ h_{n-k-1}$, then h_{n-k} preserves the families $\{N_{ix}\}$ $(i = 1, \ldots, n)$ and

$$h_{n-k}|_{N_j} = \mathrm{id}_{N_j} \ (j = 1, \ldots, n).$$

By Lemma 2, h_{n-k} is the identity on L^n, so (13) holds for f.

Case 4. We take $d_1 = d_{E_1L_1}$ such that

$$d_1|_{N_1} = g|_{N_1}.$$

Then if $h_1 = d_1^{-1} \circ g$ then h_1 is the identity on N_1 and preserves the families $\{N_{ix}\}$ $(i = 1, \ldots, n)$ by Definition 1. We take $d_2 = d_{E_2L_2}$ such that

$$d_2|_{N_2} = h_1|_{N_2}$$

etc. At the n-th step we take $d_n = d_{E_nL_n}$ such that

$$d_n|_{N_n} = h_{n-1}|_{N_n}.$$

Then $h_n = d_n^{-1} \circ h_{n-1}$ preserves the families $\{N_{ix}\}$ $(i = 1, \ldots, n)$ and is the identity on each q-line $N_1, \ldots, N_n$. By Lemma 2, h_n is the identity on L^n. Hence (14) holds for f. Theorem 3 is proved.

(4.5). Now we consider an order $C = L_1 \times \ldots \times L_k \times K$ in L^n, $n \geqslant 4$, where int $C \neq \varnothing$. Here L_j are different q-rays, issuing from the point e, and K is an $(n - k)$-dimensional quasicone with vertex e. We denote by E_i $(i = 1, \ldots, n)$ the q-hyperplane spanned by $L_1, \ldots, L_{i-1}, L_{i+1}, \ldots, L_n$, K.

THEOREM 4. Let $C = L_1 \times \ldots \times L_k \times K$, where $K \neq L \times K_1$, L is a q-ray, K_1 is a q-cone, $\dim K \geqslant 3$ be an order in L^n, $n \geqslant 4$, where ∂C does not contain a q-line, and let $f\colon L^n \to L^n$ be a C-isotonic homeomorphism. If the q-cone K lies in a horosphere,

1) provided N_1 is not a horocycle, one has

$$f = f_0 \circ d_{E_1L_1} \circ \ldots \circ d_{E_kL_k}; \tag{17}$$

2) provided N_1, N_2 are not horocycles, one has

$$f = f_0 \circ d_{E_1L_1L_2} \circ d_{E_2L_2L_1} \circ d_{E_3L_3} \circ \ldots \circ d_{E_kL_k}; \tag{18}$$

3) if $N_1, \ldots, N_t$ $(t \geqslant 3)$ are not horocycles, and $N_{t+1}, \ldots, N_k$ are horocycles, then

$$f = f_0 \circ d_{E_{t+1}L_{t+1}} \circ \ldots \circ d_{E_kL_k}; \tag{19}$$

4) if all N_i $(i = 1, \ldots, k, k \geqslant 3)$ are not horocycles, then f is quasiaffine.

If K does not lie in a horosphere,

5) if all N_i ($i = 1, \ldots, k$) are horocycles, then

$$f = f_0 \circ d_{E_1 L_1} \circ \ldots \circ d_{E_k L_k}; \tag{20}$$

6) if $N_1, \ldots, N_t$ ($t \geqslant 1$) are not horocycles, and $N_{t+1}, \ldots, N_k$ are horocycles, then

$$f = f_0 \circ d_{E_{t+1} L_{t+1}} \circ \ldots \circ d_{E_k L_k}; \tag{21}$$

7) if all N_i ($i = 1, \ldots, k$) are not horocycles, then f is quasiaffine. Everywhere here f_0 is some quasiaffine transformation and the displacements in (17)-(21) all commute. The displacements of the 1st kind are any admissible ones, and for the displacements of the 2nd kind one must consider the remark of point (4.3).

Proof. Since f is a homeomorphism, one can assume that C is closed. Each N_i containing L_i is the intersection of q-tangent q-planes to C. Under the map f, as we know, q-tangent q-planes go into q-tangent ones. Hence $f(N_i)$ will be some $N_{jf(e)}$. The quasiplane $E = \bigcap_{i=1}^{k} E_i$, spanned by the q-cone K, is mapped into the q-plane $E_{f(e)} = \bigcap_{i=1}^{k} E_{if(e)}$, since $f(E_i) = E_{jf(e)}$ due to the fact that E_i are q-tangents to C. Without loss of generality we assume that $f(e) = e$. Then $f(E) = E$ and f preserves the family of q-cones $\{K_x \colon x \in E\}$. Since $K \neq L \times K_1$, f is quasiaffine on E on the basis of Theorem 1, if E does not lie in a horosphere, and on the basis of Theorem 3 of [1], if E lies in a horosphere.

We take a q-affine bijection f_0 such that $f_0(e) = e$, $f_0(L_i) = f(L_i)$ ($i = 1, \ldots, k$), $f_0(K) = f(K)$ and f_0 coincides with f on E. Then $g = f_0^{-1} \circ f$ has the properties

$$g(C) = C,\ g(L_i) = L_i \quad (i = 1, \ldots, k),\ g|_E = \mathrm{id}_E.$$

A. Let us now assume that E is a horosphere.

Case 1. We take $d_1 = d_{E_1 L_1}$ so that d_1 coincides with g on N_1. Then $h_1 = d_1^{-1} \circ g$ preserves the order C and is the identity on N_1, E. We take $d_2 = d_{E_2 L_2}$ so that d_2 coincides with h_1 on N_2. Then $h_2 = d_2^{-1} \circ h_1$ preserves the order C and is the identity on N_1, N_2, E etc. At the k-th step we shall have $d_k = d_{E_k L_k}$ coinciding with h_{k-1} on N_k while

$$h_{k-1}|_{N_j} = \mathrm{id}_{N_j}\ (j = 1, \ldots, k-1),\ h_{k-1}|_E = \mathrm{id}_E.$$

Consequently, $h_k = d_k^{-1} \circ h_{k-1}$ will preserve $\{N_{ix} \colon x \in L^n\}$ ($j = 1, \ldots, k$), $\{E_x \colon x \in L^n\}$ and

$$h_k|_{N_j} = \mathrm{id}_{N_j} \quad (j = 1, \ldots, k),\ h_k|_E = \mathrm{id}_E.$$

From this it is easy to conclude that h_k is the identity on L^n. For this it suffices to repeat the arguments given at the end of the proof of Theorem 1. Since h_k is the identity on L^n one has that f has the form (17).

The proof of cases 2-4 is essentially a repetition of the proofs of cases 1-4 of Theorem 3. Hence we omit them.

B. Let us now assume that K does not lie in a horosphere or that E is not a horosphere.

Case 5. Let $d_1 = d_{E_1 L_1}$ be such that d_1 coincides with g on N_1 so $h_1 = d_1^{-1} \circ g$ is the identity on N_1, E, and also as before is C-isotonic. For the rest we repeat the proof of Case 1. We proceed to Cases 6 and 7.

Let N_1 not be a horocycle. We denote by Σ the quasiplane spanned by N_1, E. Obviously, $g(\Sigma) = \Sigma$. We denote the restriction of g to Σ by g_1. The map g_1 preserves on Σ the family of q-cones $\{Q_x \colon x \in \Sigma\}$, where $Q = L_1 \times K$. Just as in Theorem 1, one proves that the q-tangent quasiplanes to Q_x, $x \in \Sigma$ are mapped by q_1 to q-tangent planes. We show that there exists a family $\{\lambda_x \colon x \in \Sigma\}$ of q-lines (where $\lambda \equiv \lambda_e \subset \partial K \cup \partial K^-$, $K^- = \{y \in E \colon y \underset{K}{\leqslant} e\}$, and $\underset{K}{\leqslant}$ is the order defined by the q-cone K in E), preserved by the map g_1.

In fact, we take a q-ray $\Lambda \subset \partial K$ issuing from e, such that the q-line λ containing it is not a horocycle and along Λ the q-cone Q has q-tangent q-plane Σ_1 not lying in E. If $\Sigma_1 \cap K = \Lambda$, then $\{\lambda_x \colon x \in \Sigma\}$ is the family sought. In fact, in this case for $x \in \Sigma_1$ we have $K_x \cap \Sigma_1 = \Lambda_x$.

But g_1 preserves $\{K_x \colon x \in \Sigma\}$ and $g_1(\Sigma_1) = \Sigma_1$ since g_1 is the identity on E and g_1 maps q-tangent quasiplanes to Q_y to q-tangents. Hence Λ and Λ_x, $x \in \Sigma_1$ remain T-parallel after mapping, or more precisely, $g_1(\lambda_x) = \lambda_{g_1(x)}$, because $g_1(\lambda) = \lambda$. Now if $x \notin \Sigma_1$, then $x \in \Sigma_{1z}$ for some $z \in E$. But then $g_1(\Sigma_{1z}) = \Sigma_{1g_1(z)}$ since $g_1(\Sigma_1) = \Sigma_1$ and T-parallel q-tangent quasiplanes are mapped to T-parallels, and consequently

$$g_1(\Lambda_x) = g_1(K_x \cap \Sigma_{1z}) = g_1(K_x) \cap g_1(\Sigma_{1z}) = K_{g_1(x)} \cap \Sigma_{1g_1(z)} = K_{g_1(x)} \cap \Sigma_{1z} = \Lambda_{g_1(x)},$$

because $g_1(z) = z$, $g_1(x) \in \Sigma_{1z}$, i.e., $g_1(\lambda_x) = \lambda_{g_1(x)}$.

Let us now assume that $K \cap \Sigma_1 = K_1$ is a q-convex q-cone which does not reduce to a q-ray. In this case we consider on Σ_1 the family of q-cones $\{Q_{1x}: x \in \Sigma_1\}$, where $Q_1 = L_1 \times K_1$. We denote the restriction of g_1 to Σ_1 by g_2. Then $g_2(\Sigma_1) = \Sigma_1$. Obviously K_1 does not lie in a horosphere.

Consequently, the consideration of g_2: $\Sigma_1 \to \Sigma_1$, preserving $\{Q_{1x}: x \in \Sigma_1\}$ with the object of picking out generators of the family preserved, does not differ from the same problem which we started to solve in relation to g_1: $\Sigma \to \Sigma$ preserving $\{Q_x: x \in \Sigma\}$. In other words, the case "$K \cap \Sigma_1$ is not a q-ray" forced us to consider the same problem, but in a q-plane of lower dimension. Hence one can repeat the arguments already given above. As a result, we will introduce a quasiplane Σ_2 q-tangent to Q_1, not lying in E, preserved by g_2 and a q-ray $\Lambda_1 \subset \partial K_1$ such that the q-line λ_1 containing it is not a horocycle and Σ_2 is q-tangent to K_1 along Λ_1. If $K_1 \cap \Sigma_2 = \Lambda_1$ then the family $\{\lambda_{1x}: x \in \Sigma\}$ will be the family of q-lines sought, which is preserved under the map g_1. Here in passing from $\{\lambda_{1x}: x \in \Sigma_1\}$ to $\{\lambda_{1x}: x \in \Sigma\}$ one uses the fact that g_1 is the identity on E, as is g_2 on $E \cap \Sigma_1$. However if $K_1 \cap \Sigma_2 = K_2$ is a q-convex q-cone which does not reduce to a q-ray, then one should again lower the dimension, i.e., consider g_3, the restriction of g_2 to Σ_2 which has the properties $g_3(\Sigma_2) = \Sigma_2$, $g_3(Q_{2x}) = Q_{2g_3(x)}$ for $x \in \Sigma_2$, where $Q_2 = L_1 \times K_2$. As a result, either the required family of preserved q-lines will be picked, or we arrive at g_{m+1}: $\Sigma_m \to \Sigma_m$, $\dim \Sigma_m = 3$, g_{m+1} preserves the families of q-cones $\{Q_{mx}: x \in \Sigma_m\}$, where $Q_m = L_1 \times K_m$, $\dim K_m = 2$. In this case we take as the q-ray sought any generator of the q-cone ∂K_m (cf. the beginning of the proof of Theorem 2).

Thus, there exists a family of q-lines $\{\lambda_x: x \in \Sigma\}$, preserved by the map g_1: $\Sigma \to \Sigma$ where λ is not a horocycle. By now it is obvious, since $K \neq L \times \tilde{K}$, that one can choose two such families: $\{\lambda_x: x \in \Sigma\}$, $\{\tilde{\lambda}_x: x \in \Sigma\}$, $e \in \lambda \cap \tilde{\lambda}$, $\lambda \cap \tilde{\lambda} = \{e\}$; λ, $\tilde{\lambda}$ not horocycles, $\lambda, \tilde{\lambda} \subset \partial K \cup \partial K^-$ and $g_1(\lambda_x) = \lambda_{g_1(x)}$, $g_1(\tilde{\lambda}_x) = \tilde{\lambda}_{g_1(x)}$ for $x \in \Sigma$.

On N_1, λ, $\tilde{\lambda}$ let us span quasiplane σ. Obviously $g_1(\sigma) = \sigma$. Then by Lemma 4, g_1 will be quasiaffine on σ. But since g_1 is the identity on λ, $\tilde{\lambda}$ by virtue of the dependence of the action of g_1 on λ, $\tilde{\lambda}$, N_1 the map g_1 will be the identity on N_1. From this, thanks to the fact that g_1 is affine on N_1, we get that g_1 is the identity on N_1 or g is the identity on N_1.

The inference is as follows: g is the identity on each q-line N_j which is not a horocycle. Hence one should introduce the q-plane $\tilde{E}$ spanned on E and q-lines $N_1, \ldots, N_t$ ($t \geq 1$) which are not horocycles. Then g is the identity on $\tilde{E}$, and it remains to construct displacements d_{E_jL} ($j > t$) along the remaining q-lines $N_{t+1}, \ldots, N_k$. In Case 7 one does not have to do this, and in Case 6 one repeats the standard argumants (cf. Case 1). Theorem 4 is proved.

5. The Case $C \neq L \times K$, $\bar{C} = L \times K$

We assume that ∂C does not contain a q-line and $\operatorname{int} C \neq \varnothing$.

(5.1). On the Lobachevskii plane we consider an order defined by a q-cone C such that $\bar{C} = L_1 \times L_2$. Then C is gotten from $\bar{C}$ by subtracting one or two edges at once. In this case, as is easy to see, any C-isotonic homeomorphism can be calculated from (2), (3).

(5.2). Let C be a quasicone, defining an order in the Lobachevskii space L^3, and $\bar{C} = L_1 \times L_2 \times L_3$. Then C is obtained from $\bar{C}$ by removing edges, faces, or part of the interior of faces. In the first two cases a C-isotonic homeomorphism f is described by Theorem 3, i.e., it has one of the following forms:

1) it is quasiaffine if N_1, N_2, N_3 are not horocycles;

2) $f_0 \circ d_{E_1L_1} \circ d_{E_2L_2} \circ d_{E_3L_3}$ if only one of the q-lines N_i is not a horocycle;

3) $f_0 \circ d_{E_1L_1L_2} \circ d_{E_2L_2L_1} \circ d_{E_3L_3}$, if N_1, N_2 are not horocycles and N_3 is a horocycle. Here f_0 is a quasiaffine transformation.

If part of the interior of one face is removed, then

4) f is quasiaffine, if N_1, N_2 are not horocycles, N_3 is a horocycle, and part of a face of $L_1 \times L_2$ or $L_2 \times L_3$ is removed;

5) f is $f_0 \circ d_{E_3L_3}$ if N_1, N_2 are not horocycles, N_3 is a horocycle, and part of a face of $L_1 \times L_2$ is removed;

6) f is $f_0 \circ d_{E_1L_1}$, if N_1 is not a horocycle and N_2, N_3 are horocycles, and part of a face of $L_2 \times L_3$ is removed;

7) f is $f_0 \circ d_{E_2L_2}$ if N_1 is not a horocycle, N_2, N_3 are horocycles, and part of a face of $L_1 \times L_3$ is removed;

8) if part of the interior of two faces is removed, then f is quasiaffine.

Assertions 4-8 are trivial, since when part of the interior of a face is removed, the rest is a q-cone, for which f will preserve a generator of the boundary. Consequently, in the face, in addition to edges there appears another q-line preserved by f. It remains to apply Lemma 1 or its Euclidean analog.

(5.3). The cases of dimension 2 and 3 considered above suggest what will happen in n-dimensional space. Since $\bar{C} = L \times K$ any C-isotonic homeomorphism, being $\bar{C}$-isotonic, can be described by Theorems 3 and 4. Considering that f is a C-isotonic homeomorphism, we arrive at the following inference: the displacements in (12)-(14), (17)-(21) cannot be arbitrary but only quasiaffine, for the reasons indicated at the end of point (5.2).

Here the form (12)-(14), (17)-(21) of the map f is preserved if C is gotten from $\bar{C}$ by removing entire edges or faces of a face, etc. When one removes only part of the interior of a face (part of the interior of a face of some face of higher dimension), in the corresponding formulas for f the displacements will reduce to quasiaffine transformations. If one removes part of the interior of faces lying in horospheres, then this assertion follows from Theorem 6 of [1]. In removing part of the interior of faces not lying in horospheres, there may apppear a family of preserved q-lines $\{N_x\colon x \in L^n\}$ in addition to the q-lines $L_1, \ldots, L_k$, such that N is not a horocycle. Consequently, if among $N_1, \ldots, N_k$ there were only two which are not horocycles, N_1, N_2, then one gets three. Then by Lemma 1 f will be q-affine on N_1, N_2 and in (12), (18), displacements of the second kind disappear. If only the q-line N_1 was not a horocycle, and the rest $N_2, \ldots, N_k$ are horocycles, then in the preserved q-plane σ spanned by N_1, N there will be three families of preserved q-lines: $\{N_{1x}\}$, $\{N_x\}$ and $\{\sigma_x \cap E_{1x}\}$. In other words, f is q-affine on σ, i.e., on N_1. Hence in (14), (17) the displacement $d_{E_1L_1}$ disappears, but here displacements of the second kind do not appear, as one could think, looking at the appearance of the two preserved q-lines N_1, N which are not horocycles.

The concrete form of the C-isotonic homeomorphism f can be determined from the precise description of how C is obtained from $\bar{C}$.

6. Contingency Theorem

Let us assume that the invariant order P in L^n, $n \geqslant 2$ is a set satisfying the following condition:

A. There exists a neighborhood of the point e, such that in it, the intersection $\bar{P} \cap \bar{P}^-$ does not contain points other than e, where $P^- = \{x \in L^n\colon x \leqslant e\}$.

We show that a P-isotonic homeomorphism is necessarily C-isotonic, C being an order defined by a quasicone.

(6.1). By a quasicontingency (q-contingency) of the set $M \subset L^n$ at the point a we mean the q-cone formed by all limits of q-rays issuing from a and passing through $x \in M$, $x \neq a$, as $x \to a$. We denote the quasicontingency by qc(M, a). If the point a is not a limit for M, then by definition we shall consider that qc(M, a) = {a}. It is easy to verify that a q-contingency is a closed q-cone and $qc(M, a) = qc(\bar{M}, a)$.

Suppose given an order P on L^n. By a directed curve issuing from the point x we mean the image of the half-axis $[0, +\infty)\subset \mathbf{R}$ under a continuous and monotonic map of it into L^n under which 0 is mapped into x. Obviously any directed curve issuing from x is contained in P_x.

THEOREM 5. Let P define an order in L^n and $C = qc(P, e)$. Then

1) $C \subset \bar{P}$ and C is a closed q-convex q-cone;

2) if P is a closed set and P satisfies condition A, then the boundary ∂C does not contain q-lines and C coincides with the union F of all directed curves issuing from the point e.

Proof. By a q-ray of the q-contingency C we shall mean a q-ray issuing from e and contained in C. The case $e\notin \overline{P\backslash\{e\}}$ is trivial. We assume further that e is a limit point for P. The proof given below becomes transparent if one uses the Poincaré model.

1. Let L be a q-ray of the q-contingency C. There exist q-rays $L_n = l^+(e, x_n)$, $x_n \in P$, issuing from e, passing through x_n such that $L = \lim L_n$ as $x_n \to e$. Along with the point x_n the q-ray L_n contains all points of the form $x_{nk}=\underbrace{t_n\circ\ldots\circ t_n}_{k}(e)$, where $t_n \in T$ is a motion carrying e to x_n. As $x_n \to e$ the points x_{nk} condense on the q-rays L_n and their limits form the q-ray L. But all $x_{nk} \in P$ and consequently, $L \subset \bar{P}$. Hence $C \subset \bar{P}$. As was said above, C is trivially a closed set. We prove its q-convexity. Let L_1, L_2 be two q-rays from C. By what has been proved, $L_1, L_2 \subset \bar{P}$. Since $\bar{P}$ defines the order in L^n one has $L_{1x} \subset \bar{P}$ for any point $x \in L_2$. The set

$$\bigcup_{x\in L_2} L_{1x}\subset \overline{P},$$

as is easily seen in the Poincaré model, will contain the q-segment $[x_1, x_2]$ for any two points $x_1 \in L_1$ and $x_2 \in L_2$. By the arbitrariness of the q-rays L_1, L_2 and the points x_1, x_2 we see that the set C is q-convex.

2. Let P be closed and satisfy condition A. If ∂C contained a q-line, then in view of the q-convexity and closedness of C, $\partial C \cap \partial C^-$ would also contain a q-line. But $\partial C \subset P$ and $\partial C^- \subset P^-$. Consequently, $P \cap P^-$ would contain a q-line. The latter contradicts condition A. Thus, ∂C does not contain a q-line, so C has a strictly supporting q-plane at the point e (i.e., $|C|$ in the Poincaré model has a strictly supporting Euclidean plane in the intersection with $\{x_1 > 0\}$).

Now we show that $F \subset C$. Let us assume the contrary, i.e., that there exists a point $a \in F$, but $a \notin C$. Let L be an arc of a directed curve issuing from e and passing through a. One can include the quasicone C in a q-cone K with vertex e, which is a closed q-convex q-cone with boundary ∂K, containing no q-lines. In addition $a \notin K$. We take at the point e a strictly supporting q-plane Q, separating K from the point a. Since $C\backslash\{e\}$ lies inside K, it follows from the definition of q-contingency that there exists a neighborhood U of the point e for which $P \cap U \subset K$. Hence some initial segment of the arc L is contained in K. From this we conclude that L intersects Q. Let b be the last point (in the sense of the order on L) of the arc L at which L intersects Q. Let L' be the part of L included between b and a. Obviously $L' \subset P_b$. Since $P_b \cap U_b \subset K_b$, some initial segment of the arc L' is contained in K_b. The quasiplane Q will be strictly supporting for K_b because $b \in Q$, and under movement $e \to b$ by motions from the group T, Q goes to Q. Hence, the arc L' on the initial segment will be separated from the point a, and consequently the arc L' intersects Q in a point different from b. The latter contradicts the condition according to which this point b was chosen.

Thus, $F \subset C$. Since any q-ray from C is a directed curve, one has $C \subset F$. Thus, $C = F$. Theorem 5 is proved.

(6.2). THEOREM 6. Let $f\colon L^n \to L^n$, $n \geqslant 2$, be an isotonic homeomorphic map. Then

1) for any $x \in L^n$ we have $f(\bar{P}_x) = \bar{P}_{f(x)}$;

2) if P satisfies condition A, then $f(C_x) = C_{f(x)}$, where C is the quasicontingency of the set $\bar{P}$ at the point e, i.e., $C = qc(P, e)$.

Proof. Assertion 1 is obvious. According to Theorem 5 the quasicontingency C coincides with the union $\bar{F}$ of all directed curves in the order defined by the set $\bar{P}$. Since $f(\bar{P}_x) = \bar{P}_{f(x)}$ and f is a homeomorphism, it associates with a directed curve (in the order $\bar{P}$) another such curve. Consequently, $f(\bar{F}_x) = \bar{F}_{f(x)}$. But $C_x = \bar{F}_x$. Hence $f(C_x) = C_{f(x)}$. Theorem 6 is proved.

(6.3). A direct example of how to use Theorems 5 and 6 is the following

THEOREM 7. If P is an order of L^n, $n \geqslant 3$, and satisfies condition A, and the quasicontingency $qc(P, e) \neq L \times K$, $\operatorname{int} qc(P, e) \neq \varnothing$, then any isotonic homeomorphic map f is quasiaffine.

Proof. According to Theorem 6, $f(C_x) = C_{f(x)}$, where $C = qc(P, e)$, for any point $x \in L^n$. By Theorem 5, C does not contain q-lines, $C \neq L \times K$, so by Theorem 1 f is quasiaffine. Theorem 7 is proved.

(6.4). Remark. As follows from [3], the similarity in the description of isotonic homeomorphisms in Euclidean and hyperbolic spaces is due to a common property of the Lie algebras of an Abelian group and the group of motions T, considered in the paper. Hence, the results of the paper and also the results of A. D. Aleksandrov [1] can be recounted in the single language of the theory of ordered Lie groups.

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HARDY–LITTLEWOOD THEOREM IN DOMAINS WITH QUASICONFORMAL BOUNDARY AND ITS APPLICATIONS TO HARMONIC FUNCTIONS

M. Z. Dveirin

UDC 517.53

In the theory of functions of a complex variable the theorem of Hardy–Littlewood (cf., e.g., [1, p. 74]) on the connection between the smoothness of a function, analytic in the unit disc, and the growth of the modulus of its derivative upon approximating the boundary of the disc and also the theorem of Privalov [2] on the smoothness of conjugate harmonic functions in the disc are well known. These assertions have been generalized by a number of mathematicians [1, 3-6]. In particular, the Hardy–Littlewood and Privalov theorems have been extended to domains of the complex plane other than the disc. The latest results in this direction are due to Johnston [7], who found the analog of the Hardy–Littlewood theorem for domains with locally Lipschitz boundary, and V. A. Borodin [8], who extended Privalov's theorem to domains with piecewise-smooth boundary without null corners. In the present paper analogs of the theorems cited above are found for domains with quasiconformal boundary. With their help we prove a theorem on the rate of approximation of harmonic functions by harmonic polynomials. This question was also investigated previously; theorems on the rate of approximation of harmonic functions are due to Walsh, Sewell, and Elliott [9] (the boundary of the domain is an analytic curve), V. K. Dzyadyk [10] (smooth boundary), and V. A. Borodin [8] (piecewise-smooth boundary).

We introduce the notation and definitions needed. Let G be a simply connected finite domain with Jordan boundary L and complement Ω ; $w = \varphi(z)$ be a function which maps the domain G conformally and univalently onto $K_1 = \{w: |w| < 1\}$, where the inverse function $z = \psi(w)$ is normalized by the conditions $\psi(0)=a$, $a\in G$, $\psi'(0)>0$; $l_r=\{z: |\varphi(z)|=r\}$, $r\in(0; 1)$, is the r-th level line of the function φ; the function $w = \Phi(z)$ maps the exterior of the

Donetsk. Translated from Sibirskii Matematicheskii Zhurnal, Vol. 27, No. 3, pp. 68-73, May-June, 1986. Original article submitted July 11, 1983.

0037-4466/86/2703-0361$12.50 © 1987 Plenum Publishing Corporation