BREAKDOWN OF CONNECTEDNESS OF PHYSICAL SPACE

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UDC 530.12:531.51

The conditions under which the number of connection components of physical space changes are determined.

In this paper, we determine the conditions under which the topology of physical space changes, more precisely, becomes disconnected. This problem was investigated in [1] for a closed universe.

Let M be a connected three-dimensional Riemannian manifold with the metric $\overset{\circ}{\gamma}_{\alpha\beta}(\alpha, \beta = 1, 2, 3)$, $D_0 \subset M$ be a closed region, which is homeomorphic to a three-dimensional sphere. Let us suppose that in time $t \in [0, 1]$, the number of connection components of the manifold $M_0 = M(t = 0)$ increases, and the manifold changes into one $M_1(t = 1)$ which is no longer connected. Figuratively speaking, a region D_0 separates from M_0. So as not to complicate the presentation, we shall assume that M_1 has two connection components D_1 and C_1, i.e., $M_1 = D_1 \cup C_1$, $D_1 \cap C_1 =$. The transition from M_0 to M_1 proceeds via some critical 3-space $M_{1/2}(t = 1/2)$, which is obtained from M_0 by contracting the boundary ∂D_0 of the region D_0 to a point. Then, D_0 transforms into the region $D_{1/2}$, homeomorphic to the 3-sphere S^3. Therefore, a necessary stage in the path to separation of D_0 from M_0 is stretching M_0 along ∂D_0: the transition from M_0 to $M_{1/2}$. If $F_0 \subset M_0$ is an arbitrary closed two-dimensional submanifold intersecting D_0 along B_0 and, in addition, B_0 is homeomorphic to the 2-sphere, then at $t = 1/2$, the boundary ∂B_0 is already contracted to a point, while at $t = 1$, the region B_0 is separated from F_0. For this reason, we shall first study the breakdown of connectedness of the two-dimensional manifold F_0. We shall denote the manifold or space obtained from F_0 up to time t by F_t.

We shall realize the separation of B_0 from F_0 as follows. We shall examine the family of Riemannian metrics $a_{AB}(t)$, $t \in [0, 1]$, A, B = 1, 2, defined on the manifold F_0 and satisfying the following conditions:

1) $a_{AB}(t)$ for $0 \leqslant t < 1/2$ belongs to class C^2 and for $t \geqslant 1/2$, the first order derivatives of the functions $a_{AB}(t)$ are discontinuous on ∂B_0;

2) the length of the curve ∂B_0, calculated in the metric $a_{AB}(t)$, $t < 1/2$, approaches 0 as $t \to 1/2$ or, in other words,

$$d\sigma_t|_{\partial B_0} \underset{t\to 1/2-0}{\longrightarrow} 0, \quad d\sigma_t|_{\partial B_0} = 0 \quad \text{for} \quad t \geqslant 1/2,$$

where $d\sigma_t$ is the element of area in the metric $a_{AB}(t)$;

3) the Riemannian spaces $F_0 \setminus (B_0 \cup \partial B_0)$, $B_0 \setminus \partial B_0$ with the induced metric $a_{AB}(t)$, $t \geqslant 1/2$, supplemented with the "point" ∂B_0, are closed oriented manifolds. We shall denote them as A_t and B_t, respectively.

Let us clarify the metric conditions 1-3. We represent the transition from F_0 to F_1 through $F_{1/2}$ on the same set of points F_0. For this, the family of topologies T_t, $t \in [0, 1]$, is introduced on F_0 and, in addition, each topology T_t is matched with a topology generated by the metric $a_{AB}(t)$. Therefore, the space F_t as a set equals F_0, but, in general, it has a different topology. We can write symbolically $F_t = \langle F_0, T_t \rangle$, in particular, $B_t = \langle B_0, T_t \rangle$, having in mind the topology induced on B_t. In the topology $T_{1/2}$, the curve ∂B_0 is a point, while the calculation of the boundary $\partial_{1/2} B_{1/2}$ of the set $B_{1/2}$ in the topology $T_{1/2}$ gives $\varnothing$, i.e., $\partial_{1/2} B_{1/2} = \partial_{1/2} \langle B_0, T_{1/2} \rangle = \varnothing$, because $B_{1/2} = \langle B_0, T_{1/2} \rangle$ is already homeomorphic

Omsk State University. Translated from Izvestiya Vysshikh Uchebnykh Zavedenii, Fizika, No. 8, pp. 3-6, August, 1983. Revision submitted November 2, 1982.

to the sphere S^2. Thus condition 2 indicates that ∂B_0 is contracted into a point. The space $F_{1/2}$ is a critical space; it consists of two manifolds $A_{1/2}$ and $B_{1/2}$ with the point $< \partial B_0, T_{1/2} >$. For $t > 1/2$, the manifolds A_t and B_t represent different connection components of the separated space F_0 (the point $< \partial B_0, T_{1/2} >$ no longer represents two different points). There is nothing unnatural in this, since the connection components in reality are diffeomorphic (and isometric) to A_t and B_t, respectively. Our construction is not as convenient as the Lorentz cobordism [2] between F_0 and F_1, but, on the other hand, it is suited for comparing the integrals taken along F_t, $t < 1/2$ and F_s, $s > 1/2$, which is done below.

The constructions made above permit talking about the topological metamorphosis of the manifold F_0, due to the application of the Gauss–Bonnet theorem. This theorem says that for a two-dimensional closed oriented Riemannian manifold F of class C^2

$$\int_F \Gamma d\sigma = 2\pi\chi(F),$$

where Γ is the Gaussian curvature and $\chi(F)$ is the Euler–Poincaré characteristic.

Therefore, for $0 \leqslant t < 1/2$

$$\int_{F_0} \Gamma_t d\sigma_t = 2\pi\chi(F_0) \tag{1}$$

and for $s > 1/2$

$$\int_{A_s} \Gamma_s d\sigma_s = 2\pi\chi(A_s), \quad \int_{B_s} \Gamma_s d\sigma_s = 2\pi\chi(B_s), \tag{2}$$

where Γ_t, $d\sigma_t$ are, respectively, the Gaussian curvature and the element of area in the metric $a_{AB}(t)$. Let F_0 be homeomorphic to the sphere S^2. Then $\chi(F_0) = \chi(A_s) = \chi(B_s) = 2$. We call attention to the fact that equalities (2) were obtained as a result of condition 1, i.e., due to the loss of smoothness of the metric $a_{AB}(t)$ on ∂B_0.

Let V be a small neighborhood of the curve ∂B_0 (in the topology T_0). We shall assume that $a_{AB}(t) = a_{AB}(0)$ outside $\bar{V}$. Then, it follows from (1) and (2) that

$$\left(\int_{V\cap B_s} + \int_{V\cap A_s}\right) \Gamma_s d\sigma_s - \int_V \Gamma_t d\sigma_t = 4\pi, \text{where } t < 1/2,\ 1/2 < s$$

or

$$\int_V \left(\Gamma_s \frac{d\sigma_s}{d\sigma_t} - \Gamma_t\right) d\sigma_t = 4\pi. \tag{3}$$

Since $d\sigma_s = 0$ on ∂B_0, we obtain from (3) that there exists a neighborhood $W \subset V$ in which $\Gamma_s \gg \Gamma_t$. This means that the separation of B_0 from F_0 indicates a sharp increase in curvature.

Returning now to the breakdown of connectedness of the physical space M_0, we conclude that the separation of D_0 from M_0 is characterized by a jump in the Gaussian curvature in some neighborhood U of the "sphere" ∂D_0 for any two-dimensional closed manifold F_0 intersecting D_0. From here, we conclude that there is a jump in the scalar curvature ${}^{(3)}R$ of the manifold M_0 in some neighborhood $U \supset \partial D_0$. Indeed, ${}^{(3)}R = 2\Gamma + \varkappa$, where Γ is the Gaussian curvature of the section F_0, while $\varkappa$ is the invariant of its exterior curvature (Gauss–Codazzi equation). The section can be chosen so that $\varkappa = 0$ (for example, the section θ = const or φ = const of the closed Friedman universe). For this reason, a jump $\delta\Gamma$ in the curvature Γ implies a jump $\delta^{(3)}R$ in the curvature ${}^{(3)}R$.

Let us examine the space–time metric

$$ds^2 = (N^2 - N_\alpha N^\alpha)\, dt^2 - 2N_\alpha dt dx^\alpha - \gamma_{\alpha\beta}(x,\ t)\, dx^\alpha dx^\beta,$$

on the set of events $M_0 \times [0, 1]$ satisfying the conditions:

a) t = const is a spacelike section with metric $\gamma_{\alpha\beta}(x, t)$;

b) $\partial\gamma_{\alpha\beta}/\partial n$, where n is the normal to the section t = const, are continuous;

c) $\gamma_{\alpha\beta}(x, t) = \overset{o}{\gamma}_{\alpha\beta}$ outside some neighborhood U of the region D_0 in the topology of the manifold M_0;

d) the metrics $a_{AB}(t)$ induced on two-dimensional sections F_0 (they are induced by the metrics $\gamma_{\alpha\beta}(x, t)$) satisfy the conditions 1-3 and $d\sigma_s/d\sigma_t \leqslant 1$ for $t < 1/2$, $s > 1/2$ in U;

e) the Gaussian curvature Γ_s of the section F_0 in the metric $a_{AB}(s)$ is nonnegative ($s > 1/2$).

It follows from (3) and c-d that

$$\int_{U\cap F_0} \Gamma_s d\sigma_t \geqslant 4\pi + \int_{U\cap F_0} \Gamma_t d\sigma_t,\ t < 1/2,\ 1/2 < s$$

or

$$\langle \delta\Gamma \rangle \cdot \sigma_t(U \cap F_0) \geqslant 4\pi, \tag{4}$$

where

$$\delta\Gamma = \Gamma_s - \Gamma_t,$$

$\sigma_t(A)$ is the area of the region $A \subset F_0$ in the metric $a_{AB}(t)$, and

$$\langle f \rangle = \frac{1}{\sigma_t(A)} \int_A f d\sigma_t$$

is the integral average of the quantity f.

The dynamics of the 3-geometry is described by the Einstein equations, from which follows ([3], p. 157)

$$^{(3)}R_t + K_{2,t} = \frac{16\pi G}{c^4} \varepsilon(t),\ t \in [0,\ 1]; \tag{5}$$

$$K_{2,t} = (K_\alpha^\alpha(t))^2 - K_{\alpha\beta}(t) K^{\alpha\beta}(t),$$

where $K_{\alpha\beta}(t)$ is the tensor of the exterior curvature of the section t = const.

Then

$$\langle \delta^{(3)}R \rangle + \langle \delta K_2 \rangle = \frac{16\pi G}{c^4} \langle \delta\varepsilon \rangle, \tag{6}$$

where

$$\delta^{(3)}R = {}^{(3)}R_s - {}^{(3)}R_t,\ \delta K_2 = K_{2,s} - K_{2,t},\ \delta\varepsilon = \varepsilon(s) - \varepsilon(t),\ t < 1/2,\ 1/2 < s.$$

But, as demonstrated above,

$$\langle \delta^{(3)}R \rangle \sim 2 \langle \delta\Gamma \rangle, \tag{7}$$

where Γ is the Gaussian curvature of the two-dimensional section F_0. At the same time, due to the condition b above, the exterior curvature $K_{2,t}$ will be a continuous function on $M_0 \times [0, 1]$. Therefore

$$\langle \delta K_2 \rangle = (K_{2,s} - K_{2,t})|_{x = x_0(s,t)} \xrightarrow[\substack{t \to 1/2-0 \\ s \to 1/2+0}]{} 0. \tag{8}$$

For this reason, for some $t_0 < 1/2$, $1/2 < s_0$, the quantity $\langle \delta K_2 \rangle$ is negligibly small and then, from (4)-(8), we obtain

$$\langle \delta\varepsilon \rangle \gtrsim \frac{c^4}{2\pi G} \frac{1}{\sigma_{t_0}(U \cap F_0)}.$$

It is now entirely permissible to write

$$\langle \delta\varepsilon \rangle \gtrsim \frac{c^4}{2\pi G} \frac{1}{\sigma}, \tag{9}$$

where σ is the characteristic section of the region D_0.

Equation (9) gives us the average value of the jump in the energy density, which gives rise to separation of the region D_0.

From (9), we obtain the following estimates:

1) $\sigma \sim 10^{20}$ cm² (sun), $\langle \delta\rho \rangle = \langle \delta\varepsilon \rangle / c^2 \sim 10^7$ g/cm³;

2) $\sigma \sim 10^{12}$ cm² (neutron star), $\langle \delta\rho \rangle \sim 10^{15}$ g/cm³;

3) $\sigma \sim 10^{-66}$ cm² (singularity), $\langle \delta\rho \rangle \sim 10^{93}$ g/cm³.

Thus separation of small regions is inhibited by a strong potential barrier. Motion induced in space by a change in the topology of the space itself will require enormous expenditures of energy. The parameters of superdense configurations are close to those for separation from space. This confirms our conclusions, obtained in [1] for a closed model of the universe. Breakdown of connectedness is to be expected in gravitational collapse of massive stars because in this case singularities arise (based on Penrose's theorems [4], p. 242), which entail a singularity of the curvature. It is easy to see that the above picture of the breakdown of connectedness is in many ways similar to the process of gravitational self-closure accompanied by gravitational collapse of homogeneous spherically symmetrical configurations, analyzed in detail in [5] (p. 52). For this reason, it may be expected that singularities form due to breakdown of the connectedness of 3-space.

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THEORY OF SPATIALLY PERIODIC STRUCTURES. BOSE EXCITATION GREEN'S FUNCTIONS

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UDC 539.2:530.145

We discuss Green's function techniques in the description of spatial ordering viewed as a Bose-Einstein condensation of the density wave of the ordering units.

1. Two approaches can be used in treating spatial ordering in quantum statistics [1, 2]. The first is based on the exclusion principle, according to which units forming the spatially periodic structure (the atoms of a crystallizing liquid or solid solution or the phase separations in a quasiperiodic macrostructure of dissociating alloys) cannot occupy the same spatial position $\vec{r}$. This allows the representation of the ordering process as a redistribution of fermions over the states $\vec{r}$. The corresponding Green's function formalism is identical in form to the techniques of Gor'kov in the theory of superconductivity, and has been discussed in [1].

In the second approach, the ordering process is thought of as a redistribution of the Bose density of the ordered structure over values of the wavevector $\vec{k}$. The condition that this method be applicable is that the Bose amplitudes C_k be statistically independent for different values of $\vec{k}$ [3]. However it can easily be shown that if the total number of structural units is conserved, the C_k satisfy the relation*

$$\sum_{\kappa} \langle |C_{\kappa}|^2 \rangle = \text{const}, \tag{1}$$

*The proof of (1) is carried out in similar fashion to the case of an ordered solid solution [4], where const = C(1 − C)N, C is the concentration, and N is the total number of atoms.

Kursk Polytechnical Institute. Translated from Izvestiya Vysshikh Uchebnykh Zavedenii, Fizika, No. 8, pp. 6-11, August, 1983. Revision submitted November 22, 1982.

0038-5697/83/2608-0678$07.50