ISOTONIC MAPPINGS OF A NONCONNECTEDLY ORDERED EUCLIDEAN SPACE

A. K. Guts

UDC 513.82

We shall consider the n-dimensional Euclidean space E^n ($n \geq 2$), in which there is defined a nonconnected ordering, invariant with respect to parallel transfers.

1. We introduce an ordering geometrically in E^n, such that with each point $x \in E^n$ we associate a set $P_x \subset E^n$, with the conditions: (a) $x \in P_x$; (b) if $y \in P_x$, then $P_y \subset P_x$; and (c) for $x \neq y$ we have $P_x \neq P_y$. Then, writing the relation $y \in P_x$ as $x \leq y$, we obtain a partial ordering in E^n.

The invariance of this ordering with respect to parallel transfers is understood as follows: if t is a parallel transfer (shift) and $t(P_x)$ denotes the image of the set P_x under the transfer t, then for any point $x \in E^n$ and any shift t, we have the equation $t(P_x) = P_{t(x)}$.

Thus, an invariant ordering is defined by fixing some set P_e.

We shall fix the point e throughout this article, and shall write P instead of P_e.

Moreover, $P^- \equiv \{x \in E^n : x \leqslant e\}$.

If the set P is nonconnected, then we call the ordering nonconnected. If the ordering is connected, then the set P is connected. We call a space on which there is defined a nonconnected or a connected ordering, respectively, nonconnectedly or connectedly ordered.

The problem with which this article is connected is the study of one-to-one mappings of E^n onto itself which preserve a nonconnected invariant ordering defined in E^n. By the preservation of an ordering, we mean the following property of the mapping $f: E^n \to E^n$; for any point $x \in E^n$, we have the equation $f(P_x) = P_{f(x)}$, where P is the ordering we are considering.

Mappings with this property are called P-isotonic or simple isotonic (if this is understood to be with respect to some ordering).

2. The nonconnected ordering P which we are studying satisfies the following conditions:

A) $P = \{e\} \cup Q$, where Q is a closed connected set with interior points, not containing the point e.

B) P lies inside some convex cone with acute vertex e (an "acute vertex" means the cone does not contain a straight line).

C) There exist n rays $L_1, L_2, \ldots, L_n$, which do not lie in the same hyperplane, originating from the point e, such that for any $x \in Q$,

$$t\left(\bigcup_{i=1}^{n} L_i\right) \subset Q,$$

where t is a shift with the property $t(e) = x$.

The basic result of this article is:

<u>THEOREM A.</u> Any isotonic one-to-one mapping of the Euclidean space E^n ($n \geq 2$) onto itself is an affine transformation, excluding the special case when the ordering is defined by a quasicylinder (see Sec. 4). However, in this case we also give a complete description of the mapping.

Of course, we are assuming that the ordering satisfies conditions A), B), and C). If we consider only continuous isotonic mappings, then we need not require Q to be closed or connected.

Omsk State University, Omsk. Translated from Sibirskii Matematicheskii Zhurnal, Vol. 21, No. 3, pp. 80-88, May-June, 1980. Original article submitted June 21, 1978.

 0037-4466/80/2103-0376$07.50

3. Everywhere in this article, isotonic mappings are assumed to be homeomorphisms of E^n onto itself, since we have the following theorem of A. D. Aleksandrov:

THEOREM A'. If the ordering satisfies conditions A) and B), then any isotonic one-to-one mapping of E^n onto itself is a homeomorphism.

This theorem has not been published in this form, but its proof essentially repeats the proof of Theorem 4 in [1]. Instead of the bounded sets considered in [1], it is sufficient to consider all possible nonempty intervals $P_a \cap P_b^-$.

4. In terms of the theory of commutative topological groups, Theorem A is concerned with isomorphism of isotonic mappings (see [2]). Isotonic homeomorphisms of a connectedly ordered affine space (a commutative group) were studied by Aleksandrov [2]. An example of a connectedly ordered noncommutative Lie group, for which a theorem similar to Theorem A is true, is given in [3].

5. From the point of view of relativity theory, Theorem A means that the Lorentz group may be obtained as the corollary of a causality principle, which does not assume cause-and-effect reciprocity of events in the microcosmos.

These questions were first touched upon in [4-6].

1. Notation

The basic terminology and notation were given in the introduction.

Throughout this article, P denotes an ordering satisfying conditions A) and B).

(1.1) We denote the points of the space E^n by lower-case letters. If A is a set, then we denote by $\mathring{A}$, $\bar{A}$, and ∂A, respectively, its interior, closure and boundary.

Let M be a set in E^n. Denote by M_x the set obtained from M by using the shift t such that t(e) = x. Moreover, set $M = M_e$.

If $x, y \in E^n$, then [x, y] denotes the interval of the straight line with ends x and y, and $(x, y) = [x, y] \setminus \{x, y\}$. Finally, $|x - y|$ is the Euclidean distance between x and y.

Henceforward, $l(x, y)$ and $l^+(x, y)$ denote, respectively, the straight line passing through x and y, and the ray starting from the point x and passing through y ($x \neq y$).

If $x \in E^n$, and $r > 0$ is some number, then we denote by $B(x, r)$ the open sphere with center x and radius r.

(1.2) Since we are only interested in homeomorphisms, we may assume that $Q = \overline{(\mathring{Q})}$, since this does not reflect on Theorem A. Clearly, if P is an ordering, then $P' = \{e\} \cup \overline{(\mathring{Q})}$ is also an ordering.

(1.3) We order the points on the ray $l^+(x, y)$, where $x \neq y$, in the following natural way; we say that $v \geq w$, if $v, w \in l^+(x, y)$ and $|x - v| \geq |x - w|$.

(1.4) We note that we only need a metric for simplicity of notation, since instead of E^n we may consider an affine space.

2. Exterior Cone

(2.1) Definition 1. Set

$$C = \overline{\bigcup_{x \in \mathring{Q}} l^+(e, x)},$$

where Q is the connected part of the ordering P.

We call the cone C the exterior cone of the ordering P. Its vertex is the point e.

(2.2) If x_0 is an interior point of the set Q, then there exists a ray $l^+(v, y) \subset l^+(e, x_0)$, lying entirely inside $\mathring{Q}$. Moreover, for some $\varepsilon > 0$ the cone

$$\bigcup_{w \in \overline{B(y, \varepsilon)}} l^+(v, w)$$

lies entirely in $\mathring{Q}$ $(0 < \varepsilon < |v - y|)$.

This follows from condition A) and from the fact that P is an ordering, since if $x_0 \in \overset{\circ}{Q}$, then there exists $\delta > 0$ such that $B(x_0, \delta) \subset \overset{\circ}{Q}$. However, since $P_x \subset P$, $x \in B(x_0, \delta)$, then denoting by t the shift taking e to x_0, we obtain $B(t(x_0), 2\delta) \subset \overset{\circ}{Q}$. Since $P_{t(x)} \subset P$ for $x \in B(x_0, \delta)$, it then follows that $B(t(t(x_0)), 3\delta) \subset \overset{\circ}{Q}$ etc. There exists a natural number m such that

$$B(\underbrace{t(\ldots t}_{m}(x_0)\ldots), (m+1)\delta) \cap B(\underbrace{t(\ldots t}_{m-1}(x_0)\ldots), m\delta) \neq \varnothing.$$

Clearly repetition of this procedure will give a sphere having nonempty intersection with its predecessor. The required point v is equal to $\underbrace{t(\ldots t}_{m}(x_0)\ldots)$. The remaining part of the statement is obvious.

(2.3) LEMMA 1. $P \subset C$.

Proof. (a) If $x \in Q$ and x is the limit of the sequence $\{x_n\}$, where $x_n \in \overset{\circ}{Q}$, then the sequence of rays $\{l^+(e, x_n)\}$ converges to the ray $l^+(e, \tilde{x})$. Clearly, $x \in l^+(e, \tilde{x})$ and $l^+(e, \tilde{x}) \subset C$, i.e., $x \in C$.

(b) Let $x \in Q$, but suppose that the situation described in part (a) does not hold. Suppose that $x \notin C$. Since $P_x \subset P$, there exists a sequence of points $\{x_m\}$, $x_m \in Q \setminus \overset{\circ}{Q}$ $(m = 1, 2, \ldots)$, such that $|x_m - x_{m+1}| = |x - e|$ and $x_1 = x$. In fact, if t is a shift taking e to x, then $x_m = \underbrace{t(\ldots t}_{m}(x)\ldots)$. Clearly, $\{x_m\} \not\subset C$. Since the set C is closed, there exists $\varepsilon > 0$ such that the cone

$$K_\varepsilon = \bigcup_{v \in B(x,\varepsilon)} l^+(e, v)$$

has no common points with the cone C, besides the point e. Let $z \in \overset{\circ}{Q}$ be a point such that $\lambda = l^+(e, z) \setminus [e, z) \subset \overset{\circ}{Q}$. Set $\rho = |e - z|$. There exists a number $m_0 > 0$ such that $B(x_{m_0}, 2\rho) \subset K_\varepsilon$. Thus $t(\lambda) \cap B(x_{m_0}, 2\rho) \neq \varnothing$, where t is a shift taking e into x_{m_0}, i.e., there exists a point $y \in t(\lambda)$ such that $y \in \overset{\circ}{Q}_{x_{m_0}} \cap K_\varepsilon$. But $K_\varepsilon \cap \overset{\circ}{Q} = \varnothing$, and therefore $K_\varepsilon \cap \overset{\circ}{Q}_{x_{m_0}} = \varnothing$, which contradicts the statement we have just obtained. Thus, in fact $x \in C$.

Lemma 1 is proved.

(2.4) Clearly, we have the equation

$$\overline{\bigcup_{x \in Q} l^+(e, x)} = C.$$

(2.5) LEMMA 2. The cone C is convex.

Proof. Suppose that the statement of the lemma is false. Then there exist points $x, y \in \partial C$ such that for any $v \in (x, y)$ we have $v \notin C$. Since $C = \overline{C}$, there exists a point $v_0 \in (x, y)$ and a number $\varepsilon > 0$ such that $B(v_0, \varepsilon) \cap C = \varnothing$. Let $p', q' \in \overset{\circ}{Q}$ be points such that for points p, q lying, respectively, on the rays $l^+(e, p')$ and $l^+(e, q')$, we have

$$[p, q] \cap B(v_0, \varepsilon) \neq \varnothing. \tag{1}$$

Without loss of generality, we may assume that $p \neq p'$, $q \neq q'$

$$l^+(p, p') \subset l^+(e, p'),\ l^+(q, q') \subset l^+(e, q') \text{ and } l^+(p, p') \setminus [e, p) \subset \overset{\circ}{Q},$$
$$\lambda \equiv l^+(q, q') \setminus [e, q) \subset \overset{\circ}{Q}. \tag{2}$$

Consider the cone C_p. The ray $[l^+(q, q')]_p$ is parallel to the ray $l^+(q, q')$, and thus intersects with the cone

$$\bigcup_{w \in B(v_0,\varepsilon)} l^+(e, w). \tag{3}$$

If we now shift the cone C_p along the ray $l^+(p, p')$, we easily find a point $w \in l^+(p, p')$ such that the ray $t(\lambda)$ [see (2)], where t is a shift taking e to w, intersects with the cone (3). But $t(\lambda) \subset \overset{\circ}{Q}_w$, and the cone (3) lies outside the cone C. Thus there exists a point $\tilde{w} \in t(\lambda)$ such that $\tilde{w} \notin C$. But $\tilde{w} \in \overset{\circ}{Q}_w$, and since $P_w \subset P$, then $\tilde{w} \in \overset{\circ}{Q}$. Hence $\tilde{w} \in C$. This contradicts the above facts.

Lemma 2 is proved.

(2.6) By condition B), the exterior cone C has an acute vertex. Strictly speaking, we had the exterior cone in mind in condition B).

(2.7) It follows from condition B) that there exists a hyperplane H passing through the point e, for which we have

$$H \cap Q = \varnothing, \quad C \cap H = \{e\}.$$

We shall keep the notation H for such a hyperplane throughout this article. Moreover, let H^+ be the closed semispace generated by H containing the set P, and $H^- = E^n \setminus \overset{\circ}{H}{}^+$.

If H_x intersects with Q, then the set $H_x \cap Q$ is compact.

3. Line Ordering

(2.3) Definition 2. The ordering P is called inner-line, if there exists a ray $l^+(e, x_0) \subset \overset{\circ}{C} \cup \{e\}$, where $x_0 \in \overset{\circ}{Q}$, such that any straight line l parallel to the ray $l^+(e, x_0)$ must intersect with the set Q in a ray, i.e., if $l \parallel l^+(e, x_0)$, then $l \cap Q$ is a ray.

Definition 3. The ordering P is called boundedly line, if it is not inner-line and there exists a ray $l^+(e, x_0) \subset \partial C$, where $x_0 \in \partial C$, such that for any straight line l parallel to the ray $l^+(e, x_0)$, the set $l \cap Q$ is either empty or is a ray (i.e., $l \cap Q$ is either empty or is a ray).

The ordering P is called line if it is either inner-line or boundedly line.

In this section, we assume that the ordering P is line.

(3.2) Let

$$\widetilde{P} \equiv E^n \setminus \bigcup_{e \in \partial Q_x} \overset{\circ}{Q}_x.$$

Then we may consider the family $\{\widetilde{P}_x : x \in E^n\}$, which realizes some transfer to a connected line ordering.

LEMMA 3. If the ordering P is line,* then $\widetilde{P} \subset C^-$, where C^- denotes the cone centrally symmetric to the cone C with respect to the point e.

Proof. Let $x \in \widetilde{P}$. Four cases are possible:

1) $x \in \overset{\circ}{C}$;

2) $x \in \partial C \setminus \{e\}$;

3) $x \notin C \cup C^-$;

4) $x \in C^-$.

The fourth case leads directly to the statement of the lemma, so we shall consider the first three cases.

Since $x \in \widetilde{P}$, then for any point z such that $e \in \partial Q_z$, we have $x \notin \overset{\circ}{Q}_z$, i.e.,

$$x \notin \bigcup_{e \in \partial Q_z} \overset{\circ}{Q}_z \tag{5}$$†

1) Therefore $x \in \overset{\circ}{C}$. Let $z \in l^+(e, x)$ be the point in ∂Q most distant from e. Such a point exists, since $l^+(e, x) \subset \overset{\circ}{C} \cup \{e\}$. Moreover, this ray must pass (see Definition 1) through some point $y \in \overset{\circ}{Q}$. Thus, the ray $l^+(z, v) \subset l^+(e, x)$, where $z < v \in l^+(e, x)$ is such that $l^+(z, v) \setminus \{z\} \subset \overset{\circ}{Q}$. Let t be a shift taking the point z to e. Then $\partial t(Q) \ni e$ and $x \in t(\overset{\circ}{Q})$. But this contradicts (5).

2) Let $x \in \partial C \setminus \{e\}$. Take a point $v \in \overset{\circ}{Q}$. Then by (2.2), there exists a circular cone K with axis L_0 and vertex $w \in l^+(e, v)$, such that $K \subset \overset{\circ}{Q}$. Consider the spheres $\overline{B(z, r(z))}$, where $z \in L_0$, inscribed in K. Denote by z_0 the point such that $r(z_0) > |e - x|$.

Set $l(e, x) = l^+(e, x) \cup L$, $L \cap l^+(e, x) = \{e\}$. Then the ray L_{z_0} must intersect with ∂Q. Let $a \in L_{z_0} \cap \partial Q$ be a point such that there are no points of ∂Q on $[z_0, a)$. Clearly, $|z_0 - a| \geq r(z_0) > |e - x|$. Since $[z_0, a) \subset \overset{\circ}{Q}$, then denoting by t the shift taking a to e, we obtain

$$e \in \partial t(Q), \quad x \in t(\overset{\circ}{Q}).$$

But this contradicts (5).

*The lemma is also true for a nonline ordering.

†No Eq. (4) appears in Russian original — Publisher.

3) Let $x \notin C \cup C^-$. We use the cone K introduced in 2), and also the ray L. The ray L_{z_0} must intersect with ∂Q, since $L_{z_0} \subset H_{z_0}^-$ [see (2.7)]. Let $a \in L_{z_0}$ be the nearest point in ∂Q to z_0. Then $(a, z_0] \neq \emptyset$ and moreover, $(a, z_0] \subset \mathring{Q}$, $|a - z_0| \geqslant r(z_0) > |e - x|$.

If t is a shift such that $t(a) = e$, then

$$e \in \partial t(Q) \text{ and } x \in t(\mathring{Q}),$$

and this contradicts (5).

Lemma 3 is proved.

(3.3) In this section we shall prove an important lemma, which will be very useful.

Let Π be some unbounded set containing the point e, lying inside the convex closed cone K with acute vertex e.

LEMMA 4. If $f: E^n \to E^n$ is a homeomorphism preserving the family $\{\Pi_x : x \in E^n\}$, i.e., $f(\Pi_x) = \Pi_{f(x)}$ for any point $x \in E^n$, then there exists a set Π' containing the point e and lying inside the cone K, such that Π' defines an ordering in E^n, and moreover $f(\Pi'_x) = \Pi'_{f(x)}$, where $x \in E^n$ is an arbitrary point, i.e., f is Π'-isotonic.

Proof. Let

$$\Pi^{(0)} = \Pi, \ \Pi^{(1)} = \bigcup_{x \in \Pi} \Pi_x, \ldots, \Pi^{(n)} = \bigcup_{x \in \Pi^{(n-1)}} \Pi_x^{(n-1)}, \ldots$$

$$\Pi' = \bigcup_{n=0}^{\infty} \Pi^{(n)}.$$

We show that Π' is the required ordering. Suppose not, i.e., Π' does not define an ordering. Then there exists a point $x \in \Pi'$ such that Π'_x is not a subset of the set Π'. Therefore, there exists a point $y \in \Pi'_x$, which does not belong to Π'. Hence we see that for any n = 0, 1, . . . , the point $y \notin \Pi^{(n)}$, and at the same time there exists m_0 such that $y \in \Pi_x^{(m_0)}$. Since $x \in \Pi'$, there exists m_1 for which we have $x \in \Pi^{(m_1)}$. Thus $x \in \Pi_x^{(m_1)} \subset \Pi^{(m_1+1)}$. But it is easily seen that $\Pi^{(k)} \subset \Pi^{(k+1)}$ (k = 0, 1, . . .). Therefore,

$$y \in \Pi_x^{(m_0)} \subset \Pi_x^{(\max(m_0, m_1))},$$

$$x \in \Pi_x^{(m_1)} \subset \Pi_x^{(\max(m_0, m_1))} \subset \Pi^{(\max(m_0, m_1)+1)},$$

i.e., $y \in \Pi^{(\max(m_0, m_1)+1)}$. But this contradicts the fact that $y \notin \Pi^{(n)}$ for any n = 0, 1, ...

This contradiction shows that Π' defines an ordering on E^n. Since we constructed Π' from the family $\{\Pi_x : x \in E^n\}$, using only set-theoretical operations, clearly f preserves the ordering Π'.

Lemma 4 is proved.

Thus using Lemma 4 we may go from a given family of sets to one which defines an ordering in E^n. Moreover, the property of invariance with respect to a particular homeomorphism is preserved for this family.

(3.4) Denote by ord (Π) the set Π' in Lemma 4, obtained from the set Π, satisfying the conditions of Lemma 4.

If Π defines an ordering, then clearly ord (Π) = Π. If however Π does not define an ordering, then ord (Π) must define one.

It is also easily seen that if Π is a linearly connected set, line with respect to the ray L, then ord (Π) is line with respect to the ray L and is linearly connected.

(3.5) If P is a line ordering with respect to the ray $L_0 = l^+(e, x_0)$, then consider the following ray:

$$L_0^- = (l(e, x_0) \setminus L_0) \cup \{e\}.$$

LEMMA 5. Let P be a line ordering with respect to the ray L_0. Then there exists a linearly connected ordering P' which is line with respect to L_0^-; $L_0^- \subset P'$; and any P-isotonic homeomorphism is P'-isotonic. Moreover, $P' \subset C^-$.

Proof. (a) The set $\tilde{P}$ [see (3.2)] is line with respect to L_0^-.

In fact, let $x \in \tilde{P}$. It is sufficient to show that $L_{0x}^- \subset \tilde{P}$. Suppose not, and let $y \neq x$, $y \in L_{0x}^-$ but $y \notin \tilde{P}$. Then there exists z for which $e \in \partial Q_z$ and $y \in \mathring{Q}_z$. Let $\delta > 0$ be a number such that $B(y, \delta) \subset \mathring{Q}_z$. Since P is line with respect to L_0, then $L_{0y} \subset Q_z$. Thus $x \in Q_z$. But $x \in \tilde{P}$, i.e., there exists a point $v \in B(x, \delta/2)$, but $v \notin Q_z$. Then $v \in L_{0w}$, where $w \in B(y, \delta)$ is some point. Hence it follows that $w \in \mathring{Q}_z$, and therefore since the ordering P is line, $v \in Q_z$. Contradiction.

(b) $L_0^- \subset \tilde{P}$. In fact, since $e \in \tilde{P}$, we obtain the required result using the methods in (a).

(c) Denote by S the part of the set $\tilde{P}$ which is joined to e by some path lying in $\tilde{P}$. The set S is nonempty, by part (b), and from Lemma 3, satisfies the conditions of Lemma 4. Then ord (S) is a linearly connected ordering in E^n which is line with respect to L_0^- [see (3.4)].

Clearly, P' = ord (S) is the required ordering.

Lemma 5 is proved.

(3.6) LEMMA 6. Let the ordering P satisfy conditions A), B), and C). Then there exists a convex cone K with acute vertex e and interior points, such that any P-isotonic homeomorphism f satisfies the equation

$$f(K_x) = K_{f(x)}.$$

Proof. Let $L_1, \ldots, L_n$ be the rays in condition C). Clearly, the ordering P is line with respect to any of these rays. Set $L_i^- = (l(e, x_i) \setminus L_i) \cup \{e\}$, where $x_i \in L_i (x_i \neq e)$ is an arbitrary point on L_i.

If P' is the ordering obtained from P in Lemma 5, then P' is line with respect to any ray L_i^-. Moreover, $L_i^- \subset P'$ (i = 1, ..., n) and $P' \subset C^-$. Denote by K the contingency of the set P' at the point e. Clearly, $L_i^- \subset K$ (i = 1, ..., n). By Theorem 1a of [2], the contingency K is a convex cone with an acute vertex. Therefore, K has interior points. Since $P' \subset C^-$, P' satisfies the condition of Theorem 1 of [2]. Thus K coincides with the union of all directed curves (see [2, p. 5]) starting from e. Hence it follows that any P-isotonic homeomorphism f satisfies the equation $f(K_x) = K_{f(x)}$.

Lemma 6 is proved.

4. Proof of Theorem A

(4.1) Let E be some hyperplane, and $\mathbf{l}$ a vector (or a ray L) not parallel to E.

Definition 4. The displacement $d_{E\mathbf{l}}$ (or d_{EL}) is a mapping satisfying the following conditions:

1) $d_{E\mathbf{l}}$ (or d_{EL}) is a homeomorphism of E^n onto itself;

2) on each hyperplane parallel to E, $d_{E\mathbf{l}}$ (respectively, e_{EL}) is a shift;

3) $d_{E\mathbf{l}}$ (or d_{EL}) takes intervals (rays) equal and parallel to $\mathbf{l}$ (respectively, L) into the same intervals (rays).

Definition 5. The quasicylinder Q(E, $\mathbf{l}$) is a set M satisfying the conditions:

1) there exist hyperplanes $E_1, E_2, \ldots$, parallel to E, where E_{i+1} is obtained from E_i by a shift to the vector $\mathbf{l}$, and moreover,

$$M = \bigcup_i [M_i \cup (M \cap E_i)], \tag{6}$$

where each M_i is a cylinder formed by open intervals equal to $\mathbf{l}$ (as vectors) with ends at E_i and E_{i+1};

2) M does not admit a representation (6) with the same hyperplane E and a vector $\mathbf{l}'$ parallel to $\mathbf{l}$ but greater than $\mathbf{l}$.

The definition of the quasicylinder Q(E, L), where L is a ray, is obvious. We have taken these definitions from [2].

(4.2) THEOREM A. Let P be an ordering in E^n satisfying conditions A), B), and C). Then any P-isotonic homeomorphism $f: E^n \to E^n$ is either an affine transformation, or P is a quasicylinder and f is of the form

$$f = f_0 \circ d_1 \circ \ldots \circ d_p, \tag{7}$$

where f_0 is an affine transformation, and d_i is $d_{E_i\mathbf{l}_i}$ or $d_{E_iL_i}$. Moreover, it does not matter in which order the d_i appear.

Proof. Let P be an ordering satisfying conditions A), B), and C). By Lemma 6, any P-isotonic homeomorphism f preserves the family of sets $\{K_x : x \in E^n\}$, where K is some closed convex cone with an acute vertex.

Moreover, $\overset{\circ}{K} \neq \varnothing$. Clearly, K defines a connected ordering in E^n. Then from Theorems 3 and 4 of Aleksandrov in [2], the homeomorphism f is either an affine transformation, or may be written in the form

$$f = f_0 \circ D_1 \circ \ldots \circ D_m, \tag{8}$$

where f_0 is an affine transformation, and D_i is $d_{E_i I_i}$ or $d_{E_i L_i}$. Moreover, it does not matter in what order the displacements D_i appear. It remains to show that in the case when f is of the form (8), P is a quasicylinder and f is defined by formula (7).* However, this situation was considered in Sections 6.3-6.8 of [2] in sufficient detail. Moreover, we do not need the assumption on the connectivity of the ordering in Aleksandrov's arguments. Therefore, we see that (8) implies that P is a quasicylinder.*

The theorem is proved.

I am deeply indebted to A. D. Aleksandrov, who set the problem and greatly furthered my studies.

LITERATURE CITED

1. A. D. Aleksandrov, "A mapping of families of sets," Dokl. Akad. Nauk SSSR, 191, No. 3, 503-506 (1970).
2. A. D. Aleksandrov, "Mappings of ordered spaces," Tr. Mat. Inst. im. V. A. Steklova, 78, 3-21 (1972).
3. A. K. Guts, "Mappings of ordered Lobachevskii spaces," Dokl. Akad. Nauk SSSR, 215, No. 1, 35-37 (1974).
4. A. D. Aleksandrov and V. V. Ovchinnikova, "Remarks in relativity theory," Vestn. Leningr. Univ., 11, 95 (1953).
5. A. D. Aleksandrov, "Lorentz transformations," Usp. Mat. Nauk, 3 (37), 187 (1950).
6. E. C. Zeeman, "Causality implies the Lorentz group," J. Math. Phys., 5, No. 4, 490 (1964).

*More precisely, either f is affine, or P is a quasicylinder.

A CATEGORY OF PARTIAL RECURSIVE FUNCTIONS

A. N. Degtev

UDC 517.11:518.5

Introduction. In [1, 2], the author studied the concept of F-reducibility of one partial recursive function (p. r. f.) to another. We recall that f_0 F-reduces to f_1 ($f_0 \leq f_1$) if there exists a general recursive function (g. r. f.) g such that $f_0 = f_1 g$ (i.e., for every x either neither side of the equality is defined, or both sides are defined and equal). It was noted that this notion is closely related to m-reducibility of computable sequences of pairwise-disjoint recursively enumerable (r. e.) sets. At the same time, every p. r. f. f can be viewed as a numeration of the corresponding set $N \cup \{\omega\}$, $N = \{0, 1, \ldots\}$, where it is understood that if f(x) is undefined then in fact f(x) takes some special value ω, $\omega \notin N$. In numeration theory [3], the category of numerated sets has been studied in detail. In this paper the category $\mathscr{K}$ of p. r. f.'s compatible with F-reducibility is also considered.

We denote by rng f and dom f the range and domain of definition of f (in the usual sense). Let Rng f be rng f if dom f = N and $\operatorname{rng} f \cup \{\omega\}$ if dom f ≠ N. If $A \subseteq N$, then $\bar{A} = N \backslash A$, $f^{-1}(A) = \{x : f(x) \in A\}$, $f(A) = \{f(x) : x \in A\}$. If dom f_0 ≠ N then $\omega \in \operatorname{Rng} f_0$, and

$$f^{-1}(\operatorname{Rng} f_0) = \{x : f(x) \in \operatorname{rng} f_0 \vee f(x) = \omega\}.$$

If $A \subseteq N$, then we write |A| for the cardinality of A and $f \upharpoonright A$ for the function f_0 such that $f_0(x) = f(x)$ if $x \in A$ and $f_0(x)$ is undefined otherwise.

The objects of the category $\mathscr{K}$ are all the p. r. f.'s. If f_0 and f_1 are two objects, then the natural inclusion $\mu : \operatorname{Rng} f_0 \to \operatorname{Rng} f_1$ is called a morphism from f_0 to f_1 if there exists a g. r. f. g such that $f_0 = f_1 g$. We note that there can only exist one natural morphism from f_0 into f_1, and in order for one to exist it is necessary that $\operatorname{Rng} f_0 \subseteq \operatorname{Rng} f_1$. More precisely, the existence of a morphism from f_0 to f_1 is equivalent to F-reducibility of f_0 to f_1.

Tyumen State University, Tyumen. Translated from Sibirskii Matematicheskii Zhurnal, Vol. 21, No. 3, pp. 89-97, May-June, 1980. Original article submitted July 10, 1978.

 0037-4466/80/2103-0382$07.50